

Estimating the Shape Parameter of the Exponential-Weibull Distribution using Bayesian Technique

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Abstract

The Exponential-Weibull Distribution (EWD) has been found to be useful in modeling and prediction of real life events. Its three parameter property makes it more flexible in modeling/accommodating dataset of different characteristics. However, Bayesian technique which is more robust and efficient in most cases, has not been used to estimate the shape parameter of this very important distribution. In view of this, Bayesian technique has been employed to estimate the shape parameter of the distribution. The performance of the technique is assessed and compared with that of maximum likelihood using Monte-carlo simulation. One informative and two non-informative priors as well as three loss functions were used for the study. The results showed that; Bayesian technique produced estimators with lower MSEs regardless of the chosen sample size and parameter value.

Keywords: *Exponential-Weibull Distribution; Bayesian technique; Maximum likelihood method; Informative and non-informative priors; Loss function; Means square error*

1. Introduction

In statistics, there are two main philosophical approaches to parameter estimation. The first one is called the Classical/Frequentist approach while the second one is called the Bayesian technique. In classical or frequentist approach, the parameter(s) are considered to be fixed and known while in the Bayesian approach the parameter(s) are unknown and random (Martz and Waller (1982)). In many real life situations represented by life time models, the parameters cannot be treated as constant (fixed) throughout the life testing period (Singpurwalla, 2006) and hence the need for Bayesian estimation for life time models.

When using Bayesian method of estimation, we combine new information that is available with the prior information to form the basis for statistical inference. Statistical approaches that use prior knowledge in addition to the sample evidence to estimate the population parameter(s) are known as Bayesian methods. Specifically, this particular method seeks to optimally merge information from two sources: (1) knowledge that is known from theory or opinion formed at the beginning of the research in the form of a prior, and (2) information contained in the data in the form of likelihood functions. Basically, the prior distribution represents the initial belief of the researcher, whereas the information in the data is expressed by the likelihood function. Combining prior distribution and likelihood function, we can obtain the posterior distribution of the unknown parameter.

Bayesian technique has received increasing attention by researchers. For example, Eraikhuemen et al. (2020a), Ieren et al. (2020), Eraikhuemen et al. (2020b), Ieren and Oguntunde (2018), Preda *et al.* (2010), Dey (2010), Aliyu and Yahaya (2016), Ahmad et. al. (2016), Azam and Ahmad (2014), Dey et al. (2013), Gupta (2017), Gupta and Singh (2017) and Krishna and Goel (2017) used some prior distributions to estimate parameters of

different statistical distributions. Extensive Monte Carlo Simulations were used to compare the performances of the technique with that of Maximum Likelihood Method. It was found that; the estimates under Bayesian method have smaller MSEs compared to those obtained using the method of *MLE*. Motivated by these findings, we wish to estimate the shape parameter of the Exponential-Weibull distribution using same technique and procedure applied by the above cited authors.

2. Methods

2.1 Exponential-Weibull Distribution (EWD)

According to Bouguignon *et al.*, (2014), the cumulative distribution function (cdf) and the probability density function (pdf) (for $x > 0$) of the Exponential generator (Exp-G family) of distributions with an additional shape parameter ($\lambda > 0$) are defined by:

$$F(x) = 1 - \exp\left\{-\lambda \left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{2.1}$$

and

$$f(x) = \frac{\lambda g(x)}{[1-G(x)]^2} \exp\left\{-\lambda \left[\frac{G(x)}{1-G(x)}\right]\right\} \tag{2.2}$$

respectively, where $g(x)$ and $G(x)$ represent the pdf and the cdf of the continuous distribution to be modified, which in this case stands for the pdf and cdf of the Weibull distribution.

Substituting the pdf, $g(x)$ and cdf, $G(x)$ of the Weibull distribution in (2.1) and (2.2) above and simplifying, the cdf and pdf of the EWD for a random variable X are respectively obtained as:

$$F(x) = 1 - \exp\left\{-\lambda \left[e^{\alpha x^\beta} - 1\right]\right\} \tag{2.3}$$

and

$$f(x) = \alpha \lambda \beta x^{\beta-1} e^{\alpha x^\beta} \exp\left\{-\lambda \left[e^{\alpha x^\beta} - 1\right]\right\} \tag{2.4}$$

where $\alpha, \beta, \lambda > 0$; α is a scale parameter and β and λ are shape parameters of the EWD.

A graphical representation of the probability density function, cumulative distribution function, survival function and hazard function of the exponential-Weibull distribution (EWD) for some selected parameter values is as follows:

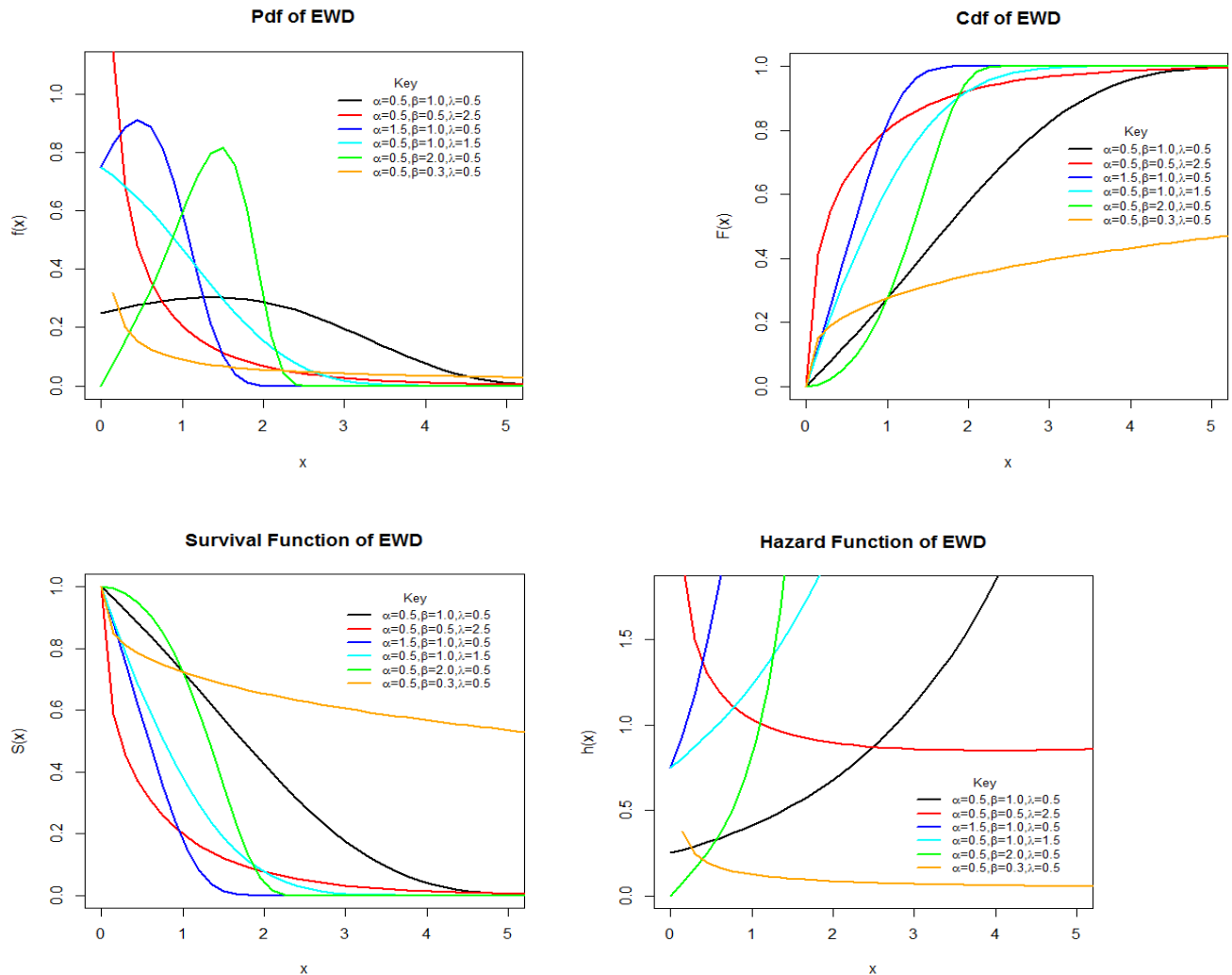


Fig. 2.1: Plots of the PDF, CDF, Survival function and Hazard function of the EWD for Selected Parameter Values.

The graphs of the pdf of the distribution in figure 2.1 above shows that it is skewed and flexible and that its shape depends on the values of the parameters. The cdf plot in figure 2.1 also shows that the distribution is a valid probability distribution.

2.2 Maximum Likelihood Estimation Method

Let x_1, x_2, \dots, x_n be a random sample from a population X with probability density function $f(x)$. The likelihood function, $L(X | \alpha, \beta, \lambda)$, is defined to be the joint density of the random variables x_1, x_2, \dots, x_n . The pdf of the exponential-Weibull distribution (EWD) is given as:

$$f(x) = \alpha\lambda\beta x^{\beta-1} e^{\alpha x^\beta} e^{-\lambda[e^{\alpha x^\beta} - 1]}$$

The likelihood function is given by;

$$L(X | \alpha, \beta, \lambda) \propto (\alpha\lambda\beta)^n \prod_{i=1}^n (x_i^{\beta-1}) e^{\alpha \sum_{i=1}^n x_i^\beta} e^{-\lambda \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1)}$$

The likelihood function for λ is given by;

$$L(X | \lambda) = \eta \lambda^n e^{-\lambda \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1)} \tag{2.5}$$

Where $\eta = (\alpha\beta)^n \prod_{i=1}^n (x_i^{\beta-1}) e^{\alpha \sum_{i=1}^n x_i^\beta}$ is a constant which is independent of the shape parameter, λ .

The log-likelihood function, $l = \log L(X | \lambda)$, is therefore given as:

$$l = \log L(X | \lambda) = n \log \lambda - \lambda \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1)$$

Differentiating l partially with respect to λ , and solving for $\hat{\lambda}$ gives;

$$\begin{aligned} \frac{\partial l}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \\ \Rightarrow \hat{\lambda} &= n \left(\sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right)^{-1} \end{aligned} \tag{2.6}$$

which is the MLE of the shape parameter λ .

2.3 Bayesian Analysis under the Assumption of Uniform Prior Using the Three Loss Functions

To obtain the posterior distribution, $P(\lambda | \underline{x})$ of the parameter, we use Bayes formula given in equation (2.7)

$$p(\lambda | \underline{x}) = \frac{p(\lambda)L(\underline{X} | \lambda)}{\int_0^\infty p(\lambda)L(\underline{X} | \lambda)d\lambda} \tag{2.7}$$

where $p(\lambda)$ is the prior distribution.

Assume that the prior distribution is uniform then,

$$p(\lambda) \propto 1; 0 < \lambda < \infty$$

using integration by substitution we obtain the posterior distribution of λ as in equation (2.8)

$$P(\lambda | \underline{x}) = \frac{\lambda^n \left[\sum_{i=1}^n (e^{\alpha x^{\beta}} - 1) \right]^{n+1}}{\Gamma(n+1) e^{\lambda \sum_{i=1}^n (e^{\alpha x^{\beta}} - 1)}} \tag{2.8}$$

We can use the squared error loss function (SELF) defined in equation (2.9) and obtain the Bayes estimator, λ_{SELF} as follows:

$$L(\lambda, \lambda_{SELF}) = (\lambda - \lambda_{SELF})^2 \tag{2.9}$$

$$\lambda_{SELF} = E(\lambda) = E(\lambda | \underline{x})$$

$$E(\lambda | \underline{x}) = \int_0^{\infty} \lambda p(\lambda | \underline{x}) d\lambda$$

$$E(\lambda | \underline{x}) = (n+1) \left[\sum_{i=1}^n (e^{\alpha x^{\beta}} - 1) \right]^{-1} \tag{2.10}$$

Similarly, quadratic loss function defined in equation (2.11) can also be used to obtain another Bayes estimator, λ_{QLF} as in equation (2.12)

$$L(\lambda, \lambda_{QLF}) = \left(\frac{\lambda - \lambda_{QLF}}{\lambda} \right)^2 \tag{2.11}$$

$$\lambda_{QLF} = \frac{E(\lambda^{-1} | \underline{x})}{E(\lambda^{-2} | \underline{x})} = \frac{\int_0^{\infty} \lambda^{-1} P(\lambda | \underline{x}) d\lambda}{\int_0^{\infty} \lambda^{-2} P(\lambda | \underline{x}) d\lambda} = \frac{(n-1)}{\left[\sum_{i=1}^n (e^{\alpha x^{\beta}} - 1) \right]} \tag{2.12}$$

In the same vein, precautionary loss function (PLF) defined in equation (2.13) has been used to derived another Bayes estimator as obtained in equation (2.14)

$$L(\lambda_{PLF}, \lambda) = \frac{(\lambda_{PLF} - \lambda)^2}{\lambda} \tag{2.13}$$

$$\lambda_{PLF} = \{E(\lambda | \underline{x})\}^{\frac{1}{2}} = \left\{ (n+1)(n+2) \left[\sum_{i=1}^n (e^{\alpha x^{\beta}} - 1) \right]^{-2} \right\}^{\frac{1}{2}}$$

$$\lambda_{PLF} = \left[(n+1)(n+2) \right]^{\frac{1}{2}} \left[\sum_{i=1}^n (e^{\alpha x^{\beta}} - 1) \right]^{-1} \tag{2.14}$$

2.4 Bayesian Analysis under the Assumption of Jeffrey’s Prior Using Three Loss Functions

The derivation of Bayes estimators continued using the three loss functions under Jeffery’s and Gamma priors. for the Jeffrey’s prior,

$$p(\lambda) \propto \frac{1}{\lambda}; 0 < \lambda < \infty \tag{2.15}$$

The posterior distribution of λ is

$$P(\lambda | \underline{x}) = \frac{\left[\sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^n \lambda^{n-1} e^{-\lambda \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1)}}{\Gamma(n)} \tag{2.16}$$

The Bayes estimators under Jeffrey’s prior using SELF, QLF and PLF are derived in equations (2.17, 2.18 and 2.19 respectively)

$$\lambda_{SELF} = \frac{\left[\sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^n}{\Gamma(n)} \int_0^\infty \lambda^n e^{-\lambda \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1)} d\lambda$$

$$\lambda_{SELF} = E(\lambda | \underline{x}) = n \left[\sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^{-1} \tag{2.17}$$

$$\lambda_{QLF} = \frac{\left[\sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]}{(n-1)} \div \frac{\left[\sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^2}{(n-1)(n-2)}$$

$$\lambda_{QLF} = \frac{(n-2)}{\sum_{i=1}^n (e^{\alpha x_i^\beta} - 1)} \tag{2.18}$$

$$\lambda_{PLF} = \{E(\lambda | \underline{x})\}^{\frac{1}{2}} = \left\{ n(n+1) \left[\sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^{-2} \right\}^{\frac{1}{2}}$$

$$\lambda_{PLF} = [n(n+1)]^{\frac{1}{2}} \left[\sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^{-1} \tag{2.19}$$

2.5 Bayesian Analysis under the Assumption of Gamma Prior Using Three Loss Functions

The gamma prior (conjugate prior) is defined as:

$$p(\lambda) = \frac{a^b}{\Gamma(b)} \lambda^{b-1} e^{-a\lambda}; a, b, \lambda > 0 \tag{2.20}$$

The posterior distribution of λ under gamma prior is derived as

$$p(\lambda | \underline{x}) = \frac{\lambda^{n+b-1} \left(a + \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right)^{n+b} e^{-\lambda \left(a + \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right)}}{\Gamma(a+b)} \tag{2.21}$$

The Bayes estimators under Gamma prior using SELF, QLF and PLF are derived as in equations (2.22, 2.23 and 2.24 respectively)

$$\lambda_{SELF} = \frac{\Gamma(n+b+1) \left[a + \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^{-1}}{\Gamma(n+b)}$$

$$\lambda_{SELF} = E(\lambda | \underline{x}) = (n+b) \left[a + \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^{-1} \tag{2.22}$$

$$\lambda_{QLF} = \frac{\left[a + \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]}{(n+b-1)} \div \frac{\left[a + \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^2}{(n+b-1)(n+b-2)}$$

$$\lambda_{QLF} = \frac{(n+b-2)}{a + \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1)} \tag{2.23}$$

$$\lambda_{PLF} = \left\{ E(\lambda | \underline{x}) \right\}^{\frac{1}{2}} = \left\{ (n+b)(n+b+1) \left[a + \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^{-2} \right\}^{\frac{1}{2}}$$

$$\lambda_{PLF} = \left[(n+b)(n+b+1) \right]^{\frac{1}{2}} \left[a + \sum_{i=1}^n (e^{\alpha x_i^\beta} - 1) \right]^{-1} \tag{2.24}$$

3 Results and Discussions

3.1 Simulation and Comparison

In this section, we compared the estimators derived in sections 2 above using simulated data from the Exponential-Weibull Distribution (EWD). Means square error (MSE) was used to assess the performance of the estimators. A random sample of size n = (15, 65, 135, 215) was generated using different parameter values.

Table 3.1: The parameter Estimates and Mean Squared Errors (MSEs) of the estimated shape parameter ($\hat{\lambda}$) for $\alpha = 0.3, \beta = 0.4, \lambda = 0.5, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes. The true shape parameter value λ equals 0.5.

n	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
15	Estimate	0.5354	0.5711	0.4997	0.5887	0.5354	0.4641	0.5530	0.5501	0.4813	0.5670
	MSE	0.0227	0.0294	0.0187	0.0338	0.0227	0.0174	0.0257	0.0233	0.0163	0.0266
65	Estimate	0.5088	0.5166	0.5009	0.5205	0.5088	0.4931	0.5127	0.5125	0.4970	0.5164
	MSE	0.0042	0.0045	0.0040	0.0047	0.0042	0.0039	0.0043	0.0043	0.0039	0.0044
135	Estimate	0.5044	0.5082	0.5007	0.5100	0.5044	0.4970	0.5063	0.5063	0.4988	0.5081
	MSE	0.0019	0.0020	0.0019	0.0021	0.0019	0.0019	0.0020	0.0020	0.0019	0.0020
215	Estimate	0.5027	0.5051	0.5004	0.5062	0.5027	0.4981	0.5039	0.5039	0.4992	0.5051
	MSE	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012

From table 3.1, we can see that the estimator using *QLF* has smaller MSE than all other estimators irrespective of the sample sizes. Hence, we can say that Bayesian estimation (using *QLF* under Uniform, Jeffrey and gamma priors) turnout to be the best.

Table 3.2: The parameter Estimates and Mean Squared Errors (MSEs) of the estimated shape parameter ($\hat{\lambda}$) for $\alpha = 0.3, \beta = 2.4, \lambda = 0.5, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes. The true shape parameter value λ equals 0.5.

N	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
15	Estimate	0.5354	0.5711	0.4997	0.5887	0.5354	0.4641	0.5530	0.5501	0.4813	0.5670
	MSE	0.0227	0.0294	0.0187	0.0338	0.0227	0.0174	0.0257	0.0233	0.0163	0.0266
65	Estimate	0.5088	0.5166	0.5009	0.5205	0.5088	0.4931	0.5127	0.5125	0.4970	0.5164
	MSE	0.0042	0.0045	0.0040	0.0047	0.0042	0.0039	0.0043	0.0043	0.0039	0.0044
135	Estimate	0.5044	0.5082	0.5007	0.5100	0.5044	0.4970	0.5063	0.5063	0.4988	0.5081
	MSE	0.0019	0.0020	0.0019	0.0021	0.0019	0.0019	0.0020	0.0020	0.0019	0.0020
215	Estimate	0.5027	0.5051	0.5004	0.5062	0.5027	0.4981	0.5039	0.5039	0.4992	0.5051
	MSE	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012

Table 3.2 also gives a similar pattern of result found in table 3.1 with the same lower values of MSE for the estimators using *QLF* under all the priors after increasing the value of β from 0.4 to 2.4. This result indicates that changing the value of β does not affect the estimate of the shape parameter, λ .

Table 3.3: The parameter Estimates and Mean Squared Errors (MSEs) of the estimated shape parameter ($\hat{\lambda}$) for $\alpha = 3.0, \beta = 0.4, \lambda = 0.5, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes. The true shape parameter value λ equals 0.5.

N	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
15	Estimate	0.5354	0.5711	0.4997	0.5887	0.5354	0.4641	0.5530	0.5501	0.4813	0.5670
	MSE	0.0227	0.0294	0.0187	0.0338	0.0227	0.0174	0.0257	0.0233	0.0163	0.0266
65	Estimate	0.5088	0.5166	0.5009	0.5205	0.5088	0.4931	0.5127	0.5125	0.4970	0.5164
	MSE	0.0042	0.0045	0.0040	0.0047	0.0042	0.0039	0.0043	0.0043	0.0039	0.0044
135	Estimate	0.5044	0.5082	0.5007	0.5100	0.5044	0.4970	0.5063	0.5063	0.4988	0.5081
	MSE	0.0019	0.0020	0.0019	0.0021	0.0019	0.0019	0.0020	0.0020	0.0019	0.0020
215	Estimate	0.5027	0.5051	0.5004	0.5062	0.5027	0.4981	0.5039	0.5039	0.4992	0.5051
	MSE	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012

Table 3.3 also gives a similar pattern of the result found in table 3.1 and 3.2 with the same lower values of MSE for the estimators using QLF under all the priors after increasing the value of α from 0.3 to 3.0. This again indicates that changing the value of α does not affect the estimates of the shape parameter, λ

Table 3.4: The parameter Estimates and Mean Squared Errors (MSEs) of the estimated shape parameter ($\hat{\lambda}$) for $\alpha = 0.3, \beta = 0.4, \lambda = 2.5, a = 1.0$ and $b = 1.0$ under three different priors and loss functions with varying sample sizes. The true shape parameter value λ equals 2.5

N	Measures	MLE	Uniform Prior			Jeffrey's Prior			Gamma Prior		
			SELF	QLF	PLF	SELF	QLF	PLF	SELF	QLF	PLF
15	Estimate	2.6772	2.8557	2.4987	2.9436	2.6772	2.3203	2.7650	2.4008	2.1007	2.4747
	MSE	0.5669	0.7358	0.4665	0.8442	0.5669	0.4345	0.6415	0.3039	0.3846	0.3131
65	Estimate	2.5438	2.5829	2.5047	2.6024	2.5438	2.4655	2.5633	2.4842	2.4090	2.5030
	MSE	0.1049	0.1131	0.0999	0.1183	0.1049	0.0980	0.1086	0.0909	0.0935	0.0921
135	Estimate	2.5222	2.5409	2.5035	2.5502	2.5222	2.4848	2.5315	2.4939	2.4573	2.5031
	MSE	0.0486	0.0505	0.0474	0.0517	0.0486	0.0470	0.0495	0.0454	0.0458	0.0457
215	Estimate	2.5137	2.5254	2.5020	2.5312	2.5137	2.4903	2.5195	2.4961	2.4730	2.5018
	MSE	0.0295	0.0303	0.0291	0.0307	0.0295	0.0289	0.0299	0.0283	0.0285	0.0284

The table 3.4 gives different outcome. The Bayes estimator using SELF under Gamma prior appeared to have outperformed all the other competing estimators for the true value of $\lambda = 2.5$. This is followed by the two Bayes estimators using PLF and QLF under the same Gamma Prior, respectively. This result is consistent across all the sample sizes.

4 Conclusion

In this study we obtain Bayesian estimators of the shape parameter of exponential-Weibull distribution (EWD). The Posterior distributions of this parameter are derived by using Uniform, Jeffrey and gamma priors. Bayes estimators and their associated mean squared errors have been derived by using three loss functions under the three prior distributions. The three loss functions taken up are Squared Error Loss Function (SELF), Quadratic Loss Function (QLF) and Precautionary Loss Function (PLF). The performances of these estimators were assessed on the basis of their mean square errors. Monte Carlo Simulations are used to compare the performance of the estimators. It is discovered that when the shape parameter is less than 1, using the *QLF* (under gamma prior) produces the lowest measures of MSE, followed by the *QLF* (under Jeffrey's prior and uniform prior) then the *SELF*, *MLE* and lastly the *PLF* under both Uniform, Jeffrey and gamma priors irrespective of the sample sizes. Most importantly, we found that Bayesian Method using quadratic Loss Function (*QLF*) under all the prior distributions results in the best estimators of the shape parameter compared to estimators using Maximum Likelihood method, Squared Error Loss Function (*SELF*) and precautionary Loss Function (*PLF*) under both Uniform and Jeffrey priors irrespective of the values of the different sample sizes. It is also discovered that the values of the other parameters have no effect on the estimators of the shape parameter of the exponential-Weibull distribution (EWD).

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