

On cardinalities of Green's equivalence classes in the semigroup of difunctional binary relations

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Abstract

In this paper, we considered the set D_X , consisting of all binary relations $\alpha \subseteq X \times X$ satisfying $(\forall x, y, u, v \in X) (x, u), (x, v), (y, u) \in \alpha \Rightarrow (y, v) \in \alpha$. This set is an inverse semigroup under a binary operation defined by $(\forall \alpha, \beta \in D_X) (x, y) \in \alpha \beta \iff x\alpha = y\beta^{-1} \neq \emptyset$, where $x\alpha$ denotes the set of images of x under α , and $y\beta^{-1}$ denotes the set of pre-images of y under β . Combinatorial results relating to Green's relations in semigroup (D_X, \diamond) are obtained. In particular, we obtained cardinalities of Green's equivalence classes in the semigroup (D_X, \diamond) for the case where X is finite. Also, we obtained the number of idempotent elements in (D_X, \diamond) to be equal to $\sum_{k=1}^n \binom{n}{k} B(k)$, where $n = |X|$ and $B(k)$ is the Bell number defined as the number of partitions of a set of k elements.

Keywords: Binary relations, Difunctional relations, Green's relations, Cardinalities

1. Introduction

One of the most natural source of examples for semigroups is the semigroup P_X of all partial transformations of a non-empty set X . This semigroup contains many interesting subsemigroups that possess certain universal properties in the theory of semigroup. For instance: the full transformation semigroup T_X (consisting of all total transformations of X) contains isomorphic copies of all semigroups; the symmetric inverse semigroup I_X (consisting of all bijections between subsets of X) contains isomorphic copies of all inverse semigroups; the symmetric group S_X (consisting of all permutations of X) contains isomorphic copies of all groups. These classes of transformation semigroups and many of their subsemigroups (such as those of order preserving and order decreasing transformations) played several important roles in both the algebraic and combinatorial aspects of semigroup theory.

The semigroups P_X , T_X , I_X and S_X are all subsemigroups, under composition, of the larger semigroup B_X of all binary relations on the set X . It is therefore natural to try to extend concepts from any of these classes of semigroups to the semigroup B_X or an interesting subsemigroup of B_X . [5] (in trying to find a natural example of inverse semigroup) defined an alternative binary operation \diamond with respect to elements of B_X , and identified an interesting subset D_X of B_X whose elements are called difunctional relations. This subset is an inverse semigroup, under \diamond , which contains the symmetric inverse semigroup I_X as a subsemigroup. Thus, the inverse semigroup (D_X, \diamond) too has the universal property of containing isomorphic copies of all inverse semigroups.

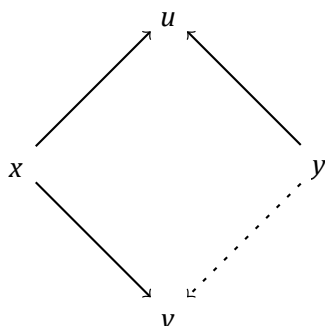
Difunctional binary relations, which were introduced in [4], appear in different areas of Discrete Mathematics, for example in [2] difunctional relations were used in databases. In this paper, we examine some combinatorial properties of the inverse semigroup $(\mathcal{D}_X, \diamond)$ of all difunctional binary relations.

2. Preliminaries

Let X be a non-empty set and define an operation \diamond on \mathcal{B}_X , the set of all binary relations on X , by the following: For any $\alpha, \beta \in \mathcal{B}_X$, $(x, y) \in \alpha \diamond \beta$ if and only if $x\alpha = y\beta^{-1} \neq \emptyset$. Then clearly, this is a binary operation on \mathcal{B}_X and so, $(\mathcal{B}_X, \diamond)$ is a groupoid. This was first defined by [5]. The operation is not associative in \mathcal{B}_X . Though the operation is not associative in \mathcal{B}_X , it still possesses some interesting properties which necessitate search for a subset of \mathcal{B}_X in which is associative. First we recall the definition of difunctional relations.

Definition 2.1: Let X be a non-empty set. A binary relation $\alpha \in \mathcal{B}_X$ is called a *difunctional* if for all $x, y, u, v \in X$,

$$(x, u), (x, v), (y, u) \in \alpha \implies (y, v) \in \alpha.$$



For example the relation $\alpha = \{(1,1), (1,3), (2,1), (2,3), (4,2), (5,2)\}$ can easily be checked as being a difunctional relation on $X = \{1,2,3,4,5,6\}$. The first connection between the operation and difunctional relations is the following observation.

Lemma 2.1: ([5]) For any $\alpha, \beta \in \mathcal{B}_X$, the product $\alpha \beta$ is a difunctional relation.

Proof. Let $\alpha, \beta \in \mathcal{B}_X$ and suppose $x, y, u, v \in X$ are such that $(x, u), (x, v), (y, u) \in \alpha \beta$. Then, by definition of the operation, it follows that $y\alpha = u\beta^{-1} = x\alpha = v\beta^{-1} \neq \emptyset$ and hence $(y, v) \in \alpha \beta$, showing that $\alpha \beta$ is a difunctional relation.

It immediately follows from this lemma that the set \mathcal{D}_X , of all difunctional binary relations on X , is closed under the operation \diamond , that is $(\mathcal{D}_X, \diamond)$ is a subgroupoid of $(\mathcal{B}_X, \diamond)$. It is already known that \mathcal{D}_X is not closed under the usual composition of binary relations. This is easily checked using simple relations, say $\alpha = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$, $\beta = \{(1,1), (2,2), (2,3), (3,2), (3,3)\}$ in \mathcal{D}_X , and noting that the composition

$$\alpha \circ \beta = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,2), (3,3)\}$$

is not in D_X since $(2,2), (2,1), (3,2) \in \alpha \circ \beta$ but $(3,1) \notin \alpha \circ \beta$. Now, since D_X is closed under \circ , the next natural question to ask is whether \circ is associative with respect to elements of D_X , even though it is not generally associative with respect to elements of B_X . This is the case as has been shown by [5]. We record the following observations about elements of D_X .

Lemma 2.2: ([5]) Let X be a non-empty set, $x, y \in X$ and $\alpha \in D_X$. Then

- (i) $x\alpha \cap y\alpha \neq \emptyset \Rightarrow x\alpha = y\alpha$.
- (ii) $x\alpha^{-1} \cap y\alpha^{-1} \neq \emptyset \Rightarrow x\alpha^{-1} = y\alpha^{-1}$.

Lemma 2.3: ([5]) Let X be a non-empty set, $x, y \in X$ and $\alpha \in D_X$. We have $(x, y) \in \alpha$ if and only if the following three conditions are satisfied:

- (i) $x\alpha \neq \emptyset$;
- (ii) $y\alpha^{-1} \neq \emptyset$;
- (iii) $x\alpha \times y\alpha^{-1} \subseteq \alpha \subseteq x\alpha \times y\alpha^{-1} \cup X \setminus x\alpha \times X \setminus y\alpha^{-1}$.

The combine effect of these lemmas is that a binary relation $\alpha \in B_X$ is difunctional if and only if there exist partitions $\{A_1, A_2, \dots\}$ and $\{B_1, B_2, \dots\}$ of $\text{dom}(\alpha)$ and $\text{im}(\alpha)$ respectively such that α is of the form

$\alpha = A_1 \times B_1 \cup A_2 \times B_2 \cup \dots$. Thus a difunctional relation $\alpha \in D_X$ can be written as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots \\ B_1 & B_2 & \dots \end{pmatrix}$$

Now, we record the lemma which proves the associativity of \circ in D_X .

Lemma 2.4: ([5]) Let $\alpha, \beta, \gamma \in D_X$ and $x, y \in X$. Then the following statements are equivalent:

- (i) $(x, y) \in (\alpha \circ \beta) \circ \gamma$,
- (ii) $(x, y) \in \alpha \circ (\beta \circ \gamma)$,
- (iii) $x\alpha \neq \emptyset, y\gamma^{-1} \neq \emptyset$, and $x\alpha \times y\gamma^{-1} \subseteq \beta \subseteq x\alpha \times y\gamma^{-1} \cup X \setminus x\alpha \times X \setminus y\gamma^{-1}$.

This together with Lemma 2.1 proves the next theorem.

Theorem 2.5: ([5]) $(\mathcal{D}_X, \diamond)$ is a semigroup.

The semigroup $(\mathcal{D}_X, \diamond)$ will be referred to as *semigroup of difunctional binary relations*. This semigroup enjoys the following properties.

Lemma 2.6: ([5]) Let $\alpha \in \mathcal{D}_X$. Then

- (i) $\alpha^{-1} \in \mathcal{D}_X$ and $\alpha \diamond \alpha^{-1} \diamond \alpha = \alpha$, that is $(\mathcal{D}_X, \diamond)$ is a regular semigroup;
- (ii) If $\alpha \diamond \alpha = \alpha$, then $\alpha^{-1} = \alpha$;
- (iii) If $\delta, \xi \in \mathcal{D}_X$ are idempotents, then $\delta \diamond \xi \subseteq \xi \diamond \delta$, and hence $\delta \diamond \xi = \xi \diamond \delta$.

The first and last parts of this lemma immediately imply the following conclusion.

Theorem 2.7: ([5]) The semigroup $(\mathcal{D}_X, \diamond)$ is an inverse semigroup.

3. Green’s Equivalence Classes

As in [1], for difunctional binary relation $\alpha = \cup_{i=1}^r A_i \times B_i = \begin{matrix} A_1 & \dots & A_r \\ B_1 & \dots & B_r \end{matrix}$ in \mathcal{D}_n , we write $\text{rank}(\alpha) = r$, $\text{dom}(\alpha) = A_1 \cup \dots \cup A_r$, $\text{codom}(\alpha) = B_1 \cup \dots \cup B_r$, $\text{ker}(\alpha) = \{A_1, \dots, A_r\}$, $\text{coker}(\alpha) = \{B_1, \dots, B_r\}$, $\text{def}(\alpha) = |X \setminus \text{dom}(\alpha)|$, $\text{codef}(\alpha) = |X \setminus \text{codom}(\alpha)|$, and call these parameters the *rank, domain, codomain, kernel, cokernel, defect* and *codefect* of α respectively. East and Vernitski [1] characterised Green’s relations on $(\mathcal{D}_n, \diamond)$ (a result which is also to be found in [3] in slightly different form).

Lemma 3.1: Let $\alpha, \beta \in \mathcal{D}_n$. Then

- (i) $\alpha L \beta$ if and only if $\text{coker}(\alpha) = \text{coker}(\beta)$,
- (ii) $\alpha R \beta$ if and only if $\text{ker}(\alpha) = \text{ker}(\beta)$,
- (iii) $\alpha J \beta$ if and only if $\text{rank}(\alpha) = \text{rank}(\beta)$,
- (iv) $\alpha H \beta$ if and only if $\text{ker}(\alpha) = \text{ker}(\beta)$ and $\text{coker}(\alpha) = \text{coker}(\beta)$,
- (v) $J = D$.

It follows from this lemma that $(\mathcal{D}_n, \diamond)$ has $n + 1$ \mathcal{J} -classes of the form $J_r = \{\alpha \in \mathcal{D}_n : \text{rank}(\alpha) = r\}$ ($0 \leq r \leq n$). Thus, the J -classes form a chain under the usual ordering:

$$J_0 < J_1 < \dots < J_n.$$

It is known that in any semigroup in which the J -classes form a chain, the ideals form a chain as well under set inclusion. So the ideals of $(\mathcal{D}_n, \diamond)$ are the sets $I_{n,r} = J_0 \cup \dots \cup J_r = \{\alpha \in \mathcal{D}_n : \text{rank}(\alpha) \leq r\}$ ($0 \leq r \leq n$). As a

consequence of the characterization of Green’s relations on $(\mathcal{D}_n, \diamond)$ presented in Lemma 3.1, we can obtain useful information about the structure of the J -classes $J_r (0 \leq r \leq n)$ in the semigroup $(\mathcal{D}_n, \diamond)$.

Theorem 3.2: Let $0 \leq r \leq n$. Then in the semigroup $(\mathcal{D}_n, \diamond)$

- (i) J_r contains $S(n + 1, r + 1)$ L-classes (and the same number of R- classes),
- (ii) each H-class in J_r contains $r!$ elements,
- (iii) each L-class in J_r contains $r!S(n + 1, r + 1)$ elements (and the same number for each R-class),
- (iv) J_r contains $S(n + 1, r + 1)^2$ H- classes,
- (v) J_r contains $r!S(n + 1, r + 1)^2$ elements,
- (vi) \mathcal{D}_n contains $\sum_{r=0}^n r!S(n + 1, r + 1)^2$ elements.

Proof. (i) The number of L-classes in J_r is obtained by [1, Lemma 2.4]. We include their description here for the sake of completeness. An L-class in J_r is uniquely determined by the cokernel of each of its elements. This is a partition $\mathbf{B} = \{B_1, \dots, B_r\}$ of some subset B of $X = \{1, 2, \dots, n\}$ for which $|B| \geq r$. The number of such partitions with $|B| = n$ is equal to $S(n, r)$ while the number of such partitions with $r \leq |B| < n$ is equal to $(r + 1)S(n, r + 1)$. Thus, the number of L-classes in J_r is equal to $(r + 1)S(n, r + 1) + S(n, r)$ and so, by applying Equation (??) we have this to be equal to $S(n + 1, r + 1)$.

Similarly, an R-class in J_r is uniquely determined by the kernel of each of its elements. This is a partition $\mathbf{A} = \{A_1, \dots, A_r\}$ of some subset A of $X = \{1, 2, \dots, n\}$ for which $|A| \geq r$. And so we arrive at the same number $S(n + 1, r + 1)$ of R-classes in J_r .

(ii) An L-class of α in J_r is completely determined by $\text{coker}(\alpha) = \{B_1, B_2, \dots, B_r\}$. Clearly from part (i) of the proposition, corresponding to this choice of $\text{coker}(\alpha)$ one can assign $S(n+1, r+1)$ possible choices of $\text{ker}(\alpha) = \{A_1, A_2, \dots, A_r\}$ and each of these $\text{ker}(\alpha)$ can be permuted in $r!$ different ways. Thus, there are in each L-class in J_r , a total of $r!S(n + 1, r + 1)$ elements. A similar observation gives the same number of elements in each R-class in J_r .

(iii) This is clear from the definition of H-classes as the intersection of L-classes and R- classes. Thus, a H-class of α in J_r is completely determined by $\text{ker}(\alpha) = \{A_1, A_2, \dots, A_r\}$ and $\text{coker}(\alpha) = \{B_1, B_2, \dots, B_r\}$. But corresponding to each choice of $\text{ker}(\alpha)$ and $\text{coker}(\alpha)$ there are clearly $r!$ maps.

(iv) Since H-classes are the cells in the egg-box picture of the J -classes J_r and there are $S(n + 1, r + 1)$ rows and $S(n + 1, r + 1)$ columns. Thus it follows that there are $S(n + 1, r + 1)^2$

cells in the J -class J_r .

(v) Clearly, from the above parts of the proposition, the size of the J -class J_r is $r!S(n+1, r + 1)^2$.

(vi) Since $|\mathcal{D}_n| = \sum_{r=0}^n |J_r|$ it follows from part (v) that $|\mathcal{D}_n| = \sum_{r=0}^n r!S(n + 1, r + 1)^2$.

Theorem 3.3: The semigroup $(\mathcal{D}_n, \diamond)$ contains $\sum_{k=1}^n \binom{n}{k} B(k)$ non-zero idempotent elements, where $B(k)$ is the Bell number.

Proof. Note that non-zero idempotent elements of rank r in $(\mathcal{D}_n, \diamond)$ are precisely those elements of the form $\xi = \cup_{i=1}^r A_i \times A_i = \begin{matrix} A_1 \dots A_r \\ A_1 \dots A_r \end{matrix}$, where $\text{dom}(\xi) = \text{codom}(\xi) = \cup_{i=1}^r A_i$ and $\text{ker}(\xi) = \text{coker}(\xi) = \{A_1, \dots, A_r\}$. Thus, the number of such non-zero idempotents of rank r in $(\mathcal{D}_n, \diamond)$ is determine by choosing a domain of size k and partitioning it into r subsets. This equals

$$\binom{n}{k} S(k, r),$$

where $S(k, r)$ is the Stirling number of the second kind. The total number of non-zero idempotents in $(\mathcal{D}_n, \diamond)$ is then realize by summing up this number over the r and over k . Thus, the number of non-zero idempotents in $(\mathcal{D}_n, \diamond)$ is

$$\begin{aligned} \sum_{k=1}^n \sum_{r=1}^k \binom{n}{k} S(k, r) &= \sum_{k=1}^n \binom{n}{k} \sum_{r=1}^k S(k, r) \\ &= \sum_{k=1}^n \binom{n}{k} B(k). \end{aligned}$$

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