

Pursuit Differential Game Problems Of L-Catch in A Hilbert Space

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Abstract

We propose some fixed duration pursuit differential game problems of l-catch in a Hilbert space. Players' motion obey ordinary differential equations and the control functions of players are subjected to either integral or geometric constraints. Pursuit is said to be completed in l-catch sense if the distance between the players with conflicting goals is less than a fixed constant l. The problem of l-catch pursuit with four different combination of player's constraints were considered and we obtained sufficient conditions for completion of pursuit in each case. Furthermore, we give examples to support our theoretical findings.

Keywords: *Differential Games, Hilbert Space, l-catch, p-strategy, and Geometric constraints*

1. Introduction

Differential game is an important area of applied mathematics with many applications. The basic differential game consists of two players, a pursuer and an evader, with conflicting goals. The motion of the pursuer and the evader is modeled by system of differential equations. Each player attempts to control the state variables of the system into a particular target set by the use of a measurable function called control, so as to achieve their goals, and the system responds to the input of both players. This game emanates from inter disciplinary research due to conflict problems during world war and its history goes back to Lion and Man problem and the Homicidal Chauffeur [20]. In recent years, differential games proved to be a tool for modeling and analyzing Economics and management problems which are characterized by both multi-period and strategic decision making and also in solving conflict between demand (predator) and supply (prey) of a certain product. Differential game can also be applied in engineering designs such as video games and robotics, especially in the aspect of programming.

A Computer game which uses a controlling device (input device) is designed such that for users to play and achieve the target of the game, they must attempt to optimize their input (which requires a lot of strategies) depending on the nature of the game. The player that responds to the users input is endowed with the ability of making several moves such that it can move to any direction in the game as instructed by the user [42]. Due to vast applications of the differential game, a lot of scholars have picked interest in game theory and optimal control where such game is been studied (see, for example, [4, 6-8, 19, 23, 26, 28, 29] and the references therein).

2. Related Works

Pursuit-Evasion differential game has numerous applications in solving real life problems and has been recently studied by many researchers (see, for example, [1, 2, 9{18,21,24,25, 32, 41, 42}]). In some of these publications (for example [9, 10, 16, 22, 41]), completion of pursuit in a finite time by the pursuer means the geometric positions of the pursuer and evader coincide. But in some works (for example [3,36, 37]), completion of pursuit means the distance between pursuer and evader is less than a fixed distance. The later meaning, is referred to as l-catch. To the best of our knowledge, there are few works in which l-catch pursuit problem was considered. Some of these can be found in [3, 22, 33{38, 40}].

In [33], Satimov and Tukhtasinov studied a pursuit and evasion problems for controlled equations of parabolic type as $z_t - A_z = -u + v, z|_t = z_0(x) z|_{st} = 0, ;$ where $z = z(x, t)$ is the unknown function, $= (x_1, x_2, \dots, x_n) \in \Omega \subset \mathbb{R}^n$. is a bounded domain, $t \in [0, T]$ with a constant positive number T. The game is described by a system of parabolic equations. Four different cases of constraints on control functions of the players were considered. In the case where the pursuer's control is subjected to the geometric constraints and evader's uses integral constraints, they showed that capture is possible in l-catch sense.

Samatov, in [37], considered problems of group pursuit in simple motion of the players for the case of l-catch, m-pursuers and one evader. The game was studied in \mathbb{R}^n and control functions of players are subjected to integral constraints and stated variables. The author analyzed that the time of catch of a group of pursuers is less than that of individual pursuer and a group of pursuers terminate the pursuit in a finite time while each of them acting individually cannot.

In [3], Badakaya examined four different pursuit differential game problems with a pursuer and an evader in the space \mathbb{R}^n . The players' motion were describe as

$$\begin{aligned} P : x(t) &= a(t)u(t); x(0) = x_0 \\ E : y(t) &= b(t)v(t); y(0) = y_0 \end{aligned}$$

Where $x_0, y_0, x, y, u, v \in \mathbb{R}^n$; $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ are the control functions of the pursuer and evader, respectively. Furthermore, $a(\cdot)$ and $b(\cdot)$ are measurable functions such that $|a(t)| \geq \delta, \delta > 0$, for all $t \in [0, \theta]$, where θ is a fixed positive number, namely the duration of the game. The player's motion obey ordinary differential equations, control functions of the players are subjected to either geometric or integral constraints, and the game is of a fixed duration. The author obtained sufficient conditions for completion of pursuit in l-catch for each of the four different problems considered.

Our objective in this paper is to propose some simple motion differential game problems of l-catch of one pursuer and one evader in the Hilbert space l_2 . The players' motion obey ordinary differential equations and the control functions of both pursuer and evader are subjected to either integral or geometric constraints. We obtain sufficient conditions for completion of pursuit in l-catch sense, and construct strategy of the pursuer that guarantees completion of pursuit in l-catch sense.

The paper is outlined as follows: In Section 2, preliminaries consisting basic definitions and lemmas that will be useful in obtaining the proposed result are given. The proposed results and supporting examples are presented in Section 3.

3. Preliminaries

Recall that a differential game problem with equations of motion of the players described by

$$\begin{cases} P : x_-(t) = u(t); x(0) = x_0; \\ E : y_-(t) = v(t); y(0) = y_0; \end{cases} \quad (1)$$

Where $x_0, y_0, x, y, u, v \in l_2$ $u(\cdot)$ and $v(\cdot)$ are the controls, which are functions of the pursuer and evader, respectively. These control functions are to satisfy one of the following inequalities;

$$\int_0^\theta \|u(t)\|^2 dt \leq \rho^2, \quad \int_0^\theta \|v(t)\|^2 dt \leq \sigma^2, \quad (2)$$

$$\|u(t)\| \leq \rho, \quad t \in [0, \theta] \quad \|v(t)\| \leq \sigma, \quad t \in [0, \theta], \quad (3)$$

$$\int_0^\theta \|u(t)\|^2 dt \leq \rho^2, \quad \|v(t)\| \leq \sigma, \quad t \in [0, \theta], \quad (4)$$

$$\|u(t)\| \leq \rho, \quad t \in [0, \theta], \quad \int_0^\theta \|v(t)\|^2 dt \leq \sigma^2, \quad (5)$$

Where ρ and σ are some positive real numbers. We shall call the equations (1), where $u(\cdot)$ and $v(\cdot)$ satisfy the inequalities in (2) (respectively, (3), (4) and (5)), game G_1 (respectively, G_2, G_3 and G_4).

Pursuit is said to be completed from the initial position x_0 of the pursuer and y_0 of the evader, if there exists strategy $u(\cdot)$ of the pursuer such that for any admissible control $v(\cdot)$ of the evader the equality $x(t) = y(t)$ is realized for some time $t \in [0, \theta]$.

We now give the criteria for the completion of pursuit with integral constraints.

Lemma 1. Let R be a fixed vector and

$$X_1 = \begin{cases} z \in l_2 : 2\langle y_0 - x_0, z \rangle \leq (\rho^2 - \sigma^2)\theta + \|y_0\|^2 - \|x_0\|^2, & \rho > \sigma, \quad x_0 \neq y_0 \\ z \in l_2 : 2\langle R, z - y_0 \rangle \leq \rho\theta, & x_0 = y_0 \end{cases}$$

If $\rho > \sigma$, $y(\theta) \in X_1$, then pursuit can be completed in the game G_1 in a finite time $t \in [0, \theta]$ provided the pursuer uses the P -strategy,

$$w(t) = \begin{cases} v(t) - \langle v(t), e \rangle e + e \sqrt{\frac{1}{2}(\rho^2 - \sigma^2) + \langle v(t), e \rangle^2}, & x_0 \neq y_0 \\ v(t), & x_0 = y_0 \end{cases} \quad (6)$$

With e the unit vector of $y_0 - x_0$.

Proof.

Let the pursuer use the P -strategy in (6). Suppose $x_0 \neq y_0$ then, we have

$$\begin{aligned} y(t) - z(t) &= y_0 - x_0 + \int_0^t v(s) ds - \int_0^t \left(v(s) - \langle v(s), e \rangle e + e \sqrt{\frac{1}{2}(\rho^2 - \sigma^2) + \langle v(s), e \rangle^2} \right) ds \\ &= \|y_0 - x_0\| e + e \int_0^t \langle v(s), e \rangle ds - e \int_0^t \sqrt{\frac{1}{2}(\rho^2 - \sigma^2) + \langle v(s), e \rangle^2} ds \\ &= ef(t), \end{aligned}$$

Where

$$f(t) = \|y_0 - x_0\| + \int_0^t \langle v(s), e \rangle ds - \int_0^t \sqrt{\frac{1}{2}(\rho^2 - \sigma^2) + \langle v(s), e \rangle^2} ds \quad (7)$$

Observe that

$$f(0) = \|y_0 - x_0\| > 0$$

We now show that $f(t) \leq 0$ to use the intermediate value theorem to show the existence of τ in the interval $[0, \theta]$ with $f(\tau) = 0$. This will imply completion of pursuit by the pursuer in a finite time. To show $f(t) \leq 0$, we consider a two dimensional vector,

$$g_1(t) = \left(\frac{1}{\sqrt{\theta}} (\rho^2 - \sigma^2)^{\frac{1}{2}}, \langle v(t), e \rangle \right).$$

Then

$$\int_0^\theta \|g_1(s)\|^2 ds = \int_0^\theta \left(\frac{1}{\theta} (\rho^2 - \sigma^2) + \langle v(s), e \rangle^2 \right) ds \tag{8}$$

And therefore

$$\begin{aligned} \left\| \int_0^\theta g_1(s) ds \right\|^2 &= \left\| \int_0^\theta \left(\frac{1}{\sqrt{\theta}} (\rho^2 - \sigma^2)^{\frac{1}{2}}, \langle v(s), e \rangle \right) ds \right\|^2 \\ &= \left\| \left(\theta^{\frac{1}{2}} (\rho^2 - \sigma^2)^{\frac{1}{2}}, \int_0^\theta \langle v(s), e \rangle ds \right) \right\|^2 \\ &= \sqrt{(\rho^2 - \sigma^2)\theta + \left(\int_0^\theta \langle v(s), e \rangle ds \right)^2}. \end{aligned} \tag{9}$$

It follows from equation (8) and (9)

$$-\int_0^\theta \left(\frac{1}{\theta} (\rho^2 - \sigma^2) + \langle v(s), e \rangle^2 \right) ds \leq -\sqrt{(\rho^2 - \sigma^2)\theta + \left(\int_0^\theta \langle v(s), e \rangle ds \right)^2}. \tag{10}$$

Therefore, from (7) and (10), we have

$$f(\theta) \leq \|y_0 - x_0\| + \int_0^\theta \langle v(s), e \rangle ds - \sqrt{(\rho^2 - \sigma^2)\theta + \left(\int_0^\theta \langle v(s), e \rangle ds \right)^2}. \tag{11}$$

Furthermore, from the definition of the set $X_1, y(\theta) \in X_1$ and for $x_0 \neq y_0$ we have

$$\begin{aligned} (\rho^2 - \sigma^2)\theta + \|y_0\|^2 - \|x_0\|^2 &\geq 2\langle y_0 - x_0, y(\theta) \rangle \\ &= 2\|y_0 - x_0\| \langle e, y(\theta) \rangle \end{aligned}$$

This implies that

$$\langle e, y(\theta) \rangle \leq \frac{(\rho^2 - \sigma^2)\theta + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|}$$

Consequently we have,

$$\begin{aligned} \frac{(\rho^2 - \sigma^2)\theta + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|} &\geq \langle e, y_0 + \int_0^\theta v(s) ds \rangle \\ &= \langle e, \int_0^\theta v(s) ds \rangle + \langle e, y_0 \rangle \end{aligned}$$

From which we deduce that

$$\int_0^\theta \langle e, y(\theta) \rangle \leq \frac{(\rho^2 - \sigma^2)\theta + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|} + \langle e, y_0 \rangle \tag{12}$$

Substituting (10) and (12) we obtain

$$f(\theta) \leq \|y_0 - x_0\| + d_1 - \langle e, y_0 \rangle - \sqrt{(\rho^2 - \sigma^2)\theta + (d - \langle e, y_0 \rangle)^2}, \tag{13}$$

Where

$$d_1 = \frac{(\rho^2 - \sigma^2)\theta + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|}$$

We now show that the right hand side of the inequality above is zero. This is true if and only if

$$\|y_0 - x_0\| + d_1 - \langle e, y_0 \rangle = \sqrt{(\rho^2 - \sigma^2)\theta + (d - \langle e, y_0 \rangle)^2}, \tag{14}$$

By squaring both side of (14) we obtain

$$(\|y_0 - x_0\| + d_1 - \langle e, y_0 \rangle)^2 = \beta + (d - \langle e, y_0 \rangle)^2,$$

Where $\beta = (\rho^2 - \sigma^2)\theta$. Now using d_1 , we have

$$\begin{aligned} \|y_0 - x_0\|^2 + 2\|y_0 - x_0\|d - \langle e, y_0 \rangle &= \|y_0 - x_0\|^2 + 2\|y_0 - x_0\| \left(\frac{\beta + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|} - \langle e, y_0 \rangle \right) \\ &= \|y_0 - x_0\|^2 + \beta + \|y_0\|^2 - \|x_0\|^2 - 2\|y_0 - x_0\|\langle e, y_0 \rangle \\ &= \|y_0 - x_0\|^2 + \beta + \|y_0\|^2 - \|x_0\|^2 - 2\|y_0 - x_0\| \left\langle \frac{y_0 - x_0}{\|y_0 - x_0\|}, y_0 \right\rangle \\ &= \|y_0 - x_0\|^2 + \beta + \|y_0\|^2 - \|x_0\|^2 - 2\langle y_0 - x_0, y_0 \rangle \\ &= \|y_0 - x_0\|^2 + \beta + \|y_0\|^2 - \|x_0\|^2 - 2\|y_0\|^2 + 2\langle x_0, y_0 \rangle \\ &= \beta \end{aligned}$$

This means

$$(\|y_0 - x_0\| + d_1 - \langle e, y_0 \rangle)^2 = (\rho^2 - \sigma^2)\theta + (d - \langle e, y_0 \rangle)^2$$

This means that the right hand side of the inequality (21) is equal to zero, that is $f(\theta) \leq 0$

It shows that there exist $\tau \in [0, \theta]$ such that $f(\tau) = 0$

$$y(\tau) - x(\tau) = ef(\tau) = 0$$

This implies that $y(\tau) = x(\tau)$, for some $\tau \in [0, \theta]$

If $x_0 = y_0$ then the pursuer uses the strategy $u(t) = v(t)$, and we have,

$$x(\theta) = x_0 + \int_0^\theta u(s)ds = y_0 + \int_0^\theta v(s)ds = y(\theta)$$

This complete the proof.

We then analyze the criteria for the completion of pursuit with geometric constraints.

Lemma 2. Let R be a fixed vector and let

$$X_2 = \begin{cases} z \in l_2 : 2\langle y_0 - x_0, z \rangle \leq (\rho^2 - \sigma^2)\theta^2 + \|y_0\|^2 - \|x_0\|^2, & \rho > \sigma, x_0 \neq y_0 \\ z \in l_2 : 2\langle R, z - y_0 \rangle \leq \rho\theta, & x_0 = y_0 \end{cases}$$

If $\rho > \sigma$, $y(\theta) \in X_2$, then pursuit can be completed in the game G_2 in a finite time $t \in [0, \theta]$ is possible provided the pursuer uses the P -strategy,

$$w(t) = \begin{cases} v(t) - \langle v(t), e \rangle e + e\sqrt{(\rho^2 - \sigma^2) + \langle v(t), e \rangle^2}, & x_0 \neq y_0 \\ v(t), & x_0 = y_0 \end{cases} \tag{15}$$

With e the unit vector of $y_0 - x_0$

The proof follows similar lines with the proof of Lemma 1. However, for completeness we give the proof.

Proof.

Let the pursuer use the P-strategy in (15). Suppose $x_0 = y_0$ then, we have

$$\begin{aligned} y(t) - x(t) &= y_0 - x_0 + \int_0^t v(s) ds - \int_0^t \left(v(s) - \langle v(s), e \rangle e + e \sqrt{(\rho^2 - \sigma^2) + \langle v(s), e \rangle^2} \right) ds \\ &= \|y_0 - x_0\| e + e \int_0^t \langle v(s), e \rangle ds - e \int_0^t \sqrt{(\rho^2 - \sigma^2) + \langle v(s), e \rangle^2} ds \\ &= ef(t), \end{aligned}$$

Where

$$f(t) = \|y_0 - x_0\| + \int_0^t \langle v(s), e \rangle ds - \int_0^t \sqrt{(\rho^2 - \sigma^2) + \langle v(s), e \rangle^2} ds$$

Observe that

$$f(0) = \|y_0 - x_0\| > 0$$

We now show that $f(t) \leq 0$ to use the intermediate value theorem to show the existence of τ in the interval $[0, \theta]$ with $f(\tau) = 0$. This will imply completion of pursuit by the pursuer in a finite time. To show $f(t) \leq 0$, we consider a two dimensional vector,

$$g_2(t) = \left((\rho^2 - \sigma^2)^{\frac{1}{2}}, \langle v(t), e \rangle \right).$$

Then

$$\int_0^\theta \|g_2(s)\| ds = \int_0^\theta \sqrt{(\rho^2 - \sigma^2) + \langle v(s), e \rangle^2} ds \tag{17}$$

And therefore

$$\begin{aligned} \left\| \int_0^\theta g_2(s) ds \right\| &= \left\| \int_0^\theta \left((\rho^2 - \sigma^2)^{\frac{1}{2}}, \langle v(s), e \rangle \right) ds \right\| \\ &= \left\| ((\rho^2 - \sigma^2)^{\frac{1}{2}} \theta, \int_0^\theta \langle v(s), e \rangle ds) \right\| \\ &= \sqrt{(\rho^2 - \sigma^2) \theta^2 + \left(\int_0^\theta \langle v(s), e \rangle ds \right)^2}. \end{aligned} \tag{18}$$

It follows from equation (8) and (9)

$$- \int_0^\theta \sqrt{(\rho^2 - \sigma^2) + \langle v(s), e \rangle^2} ds \leq - \sqrt{(\rho^2 - \sigma^2) \theta^2 + \left(\int_0^\theta \langle v(s), e \rangle ds \right)^2}. \tag{19}$$

Therefore, from (7) and (10), we have

$$f(\theta) \leq \|y_0 - x_0\| + \int_0^\theta \langle v(s), e \rangle ds - \sqrt{(\rho^2 - \sigma^2) \theta^2 + \left(\int_0^\theta \langle v(s), e \rangle ds \right)^2}. \tag{20}$$

Furthermore, from the definition of the set $X_2, y(\theta) \in X_2$ and for $x_0 \neq y_0$ we have

$$(\rho^2 - \sigma^2)\theta^2 + \|y_0\|^2 - \|x_0\|^2 \geq 2\langle y_0 - x_0, y(\theta) \rangle = 2\|y_0 - x_0\|\langle e, y(\theta) \rangle$$

This implies that

$$\langle e, y(\theta) \rangle \leq \frac{(\rho^2 - \sigma^2)\theta^2 + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|}$$

Consequently we have,

$$\begin{aligned} \frac{(\rho^2 - \sigma^2)\theta^2 + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|} &\geq \langle e, y_0 + \int_0^\theta v(s)ds \rangle \\ &= \langle e, \int_0^\theta v(s)ds \rangle + \langle e, y_0 \rangle \end{aligned}$$

From which we deduce that

$$\int_0^\theta \langle e, y(\theta) \rangle \leq \frac{(\rho^2 - \sigma^2)\theta^2 + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|} + \langle e, y_0 \rangle \tag{21}$$

Substituting (19) and (21) we obtain

$$f(\theta) \leq \|y_0 - x_0\| + d_2 - \langle e, y_0 \rangle - \sqrt{(\rho^2 - \sigma^2)\theta^2 + (d - \langle e, y_0 \rangle)^2}, \tag{22}$$

Where

$$d_2 = \frac{(\rho^2 - \sigma^2)\theta^2 + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|}$$

We now show that the right hand side of the inequality above is zero. This is true if and only if

$$\|y_0 - x_0\| + d_2 - \langle e, y_0 \rangle = \sqrt{(\rho^2 - \sigma^2)\theta^2 + (d - \langle e, y_0 \rangle)^2}, \tag{23}$$

By squaring both side of (14) we obtain

$$(\|y_0 - x_0\| + d - \langle e, y_0 \rangle)^2 = \gamma + (d - \langle e, y_0 \rangle)^2, \tag{24}$$

Where $\beta = (\rho^2 - \sigma^2)\theta^2$, Now using d , we have

$$\begin{aligned} \|y_0 - x_0\|^2 + 2\|y_0 - x_0\|d_2 - \langle e, y_0 \rangle &= \|y_0 - x_0\|^2 + 2\|y_0 - x_0\|\left(\frac{\gamma + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|} - \langle e, y_0 \rangle\right) \\ &= \|y_0 - x_0\|^2 + \gamma + \|y_0\|^2 - \|x_0\|^2 - 2\|y_0 - x_0\|\langle e, y_0 \rangle \\ &= \|y_0 - x_0\|^2 + \gamma + \|y_0\|^2 - \|x_0\|^2 - 2\|y_0 - x_0\|\left\langle \frac{y_0 - x_0}{\|y_0 - x_0\|}, y_0 \right\rangle \\ &= \|y_0 - x_0\|^2 + \gamma + \|y_0\|^2 - \|x_0\|^2 - 2\langle y_0 - x_0, y_0 \rangle \\ &= \|y_0 - x_0\|^2 + \gamma + \|y_0\|^2 - \|x_0\|^2 - 2\|y_0\|^2 + 2\langle x_0, y_0 \rangle \\ &= \gamma \end{aligned}$$

This means

$$(\|y_0 - x_0\| + d_2 - \langle e, y_0 \rangle)^2 = (\rho^2 - \sigma^2)\theta^2 + (d - \langle e, y_0 \rangle)^2$$

This means that the right hand side of the inequality (21) is equal to zero, that is $f(\theta) \leq 0$

It shows that there exist $\tau \in [0, \theta]$ such that $f(\tau) = 0$

$$y(\tau) - x(\tau) = ef(\tau) = 0$$

This implies that $y(\tau) = x(\tau)$, for some $\tau \in [0, \theta]$

If $x_0 = y_0$ then the pursuer uses the strategy $u(t) = v(t)$, and we have,

$$x(\theta) = x_0 + \int_0^\theta u(s)ds = y_0 + \int_0^\theta v(s)ds = y(\theta)$$

This complete the proof of the Lemma.

We now recall the following definition which will be needed in next section.

Definition 1. Pursuit is said to be completed in the game G_1 (respectively, G_2 , G_3 , and G_4), In the l -catch sense, if for any admissible control $v(t)$ of the evader there exists pursuers Strategy $u(t)$ such that $\|x(\theta) - y(\theta)\| \leq l$ where l is a positive number.

4. Main Result

This section contains two theorems, one of the theorem provides solution to game G_1 and G_3 , the second theorem provides solution to the game G_2 and G_4 . It also contains some illustrative examples for implementations of the theorems.

Theorem 1. If $\|x_0 - y_0\| + \sqrt{\theta}\sigma \leq \sqrt{\theta}\rho + l$, for some time θ , then pursuit can be completed in the l -catch sense in the game G_1 and G_3

Proof. Consider a dummy pursuer whose equation of motion is described by,

$$P : z(t) = w(t), \quad z(0) = x_0$$

for which the control $w(t)$ is such that

$$\int_0^\theta \|w(t)\|^2 dt \leq \bar{\rho}^2,$$

Where

$$\bar{\rho} = \rho + \frac{1}{\sqrt{\theta}} \tag{25}$$

By using the inequality in the hypothesis of the theorem we can show that $\bar{\rho} > \sigma$ Indeed,

$$\bar{\rho} = \rho + \frac{1}{\sqrt{\theta}} \geq \frac{\|x_0 - y_0\|}{\sqrt{\theta}} + \sigma > \sigma$$

Thus,

$$\theta \geq \frac{\|x_0 - y_0\|^2}{(\bar{\rho} - \sigma)^2}$$

Which is equivalent to

$$\|x_0 - y_0\|^2 \leq (\bar{\rho} - \sigma)^2 \theta$$

Moreover

$$\begin{aligned} \|x_0 - y_0\|^2 &\leq (\bar{\rho} - \sigma)^2 \theta \\ \|x_0 - y_0\| + \sigma\sqrt{\theta} &\leq \bar{\rho}\sqrt{\theta} \\ \|x_0 - y_0\| + \sigma\sqrt{\theta} &\leq \sqrt{\theta}\rho + l \end{aligned}$$

The last inequality holds from the hypothesis of the theorem.

If the dummy pursuer uses the P-strategy () the equality $y(\theta) = z(\theta)$ can be established as proved in Lemma 1. We now construct the strategy of the real pursuer as

$$u(t) = \frac{\rho}{\bar{\rho}} w(t), \quad t \in [0, \theta] \tag{26}$$

The admissibility of this strategy with respect to the integral constraints can be seen from the following

$$\begin{aligned} \int_0^\theta |u(t)|^2 dt &= \int_0^\theta \left| \frac{\rho}{\bar{\rho}} w(t) \right|^2 dt \\ &= \left| \frac{\rho}{\bar{\rho}} \right|^2 \int_0^\theta |w(t)|^2 dt \\ &\leq \rho^2 \end{aligned}$$

We now show that this strategy ensures completion of pursuit of the game by the real pursuer in l-catch sense. Indeed

$$\begin{aligned} |x(\theta) - y(\theta)| &= |x(\theta) - z(\theta)| \\ &= \left| x_0 + \int_0^\theta u(t) dt - x_0 - \int_0^\theta w(t) dt \right| \\ &= \left| \int_0^\theta u(t) dt - \int_0^\theta w(t) dt \right| \\ &= \left| \int_0^\theta \frac{\rho}{\bar{\rho}} w(t) dt - \int_0^\theta w(t) dt \right| \\ &= \left| \frac{\rho}{\bar{\rho}} - 1 \right| \left| \int_0^\theta w(t) dt \right| \\ &\leq \frac{l}{\rho\sqrt{\theta}} \left(\int_0^\theta dt \right)^2 \left(\int_0^\theta |w|^2 dt \right)^2 \\ &\leq \frac{l}{\rho\sqrt{\theta}} \sqrt{\theta} (\bar{\rho}^2)^{\frac{1}{2}} \end{aligned}$$

Hence the strategy constructed ensures completion of pursuit in the game g1 and g3 in l-catch sense.

Theorem 2. If $\|x_0 - y_0\| + \theta\sigma \leq \theta\rho + l$, for some time θ , then pursuit can be completed in the l-catch sense in the game G_2 and G_4

Proof. Consider a dummy pursuer whose equation of motion is described by,

$$P : z(t) = w(t), \quad z(0) = x_0$$

For which the control $w(t)$ is such that

$$\|w(t)\| \leq \bar{\rho}, \quad t \in [0, \theta]$$

With

$$\bar{\rho} = \rho + \frac{1}{\theta}$$

By using the inequality in the hypothesis of the theorem we can show that $\bar{\rho} > \sigma$ Indeed,

$$\bar{\rho} = \rho + \frac{1}{\theta} \geq \frac{\|x_0 - y_0\|}{\theta} + \sigma > \sigma$$

Thus,

$$\theta \geq \frac{\|x_0 - y_0\|}{(\bar{\rho} - \sigma)}$$

Which is equivalent to

$$\|x_0 - y_0\| \leq (\bar{\rho} - \sigma)\theta$$

Moreover

$$\begin{aligned} |x_0 - y_0| &\leq (\bar{\rho} - \sigma)\theta \\ |x_0 - y_0| + \sigma\theta &\leq \bar{\rho}\theta \\ |x_0 - y_0| + \sigma\theta &\leq \theta\rho + l \end{aligned}$$

The last inequality holds from the hypothesis of the theorem.

If the dummy pursuer uses the P-strategy (15), the equality $y(\theta) = z(\theta)$ can be established as proved in Lemma 2. We now consider the strategy of the real pursuer as (26) and

$$\bar{\rho} = \rho + \frac{1}{\theta}$$

With

$$\theta \geq \frac{|x_0 - y_0|}{(\bar{\rho} - \sigma)}$$

Then completion of pursuit in l-catch sense is possible in the game G_2 and G_4 since the following inequality holds:

$$\begin{aligned} |x(\theta) - y(\theta)| &= |x(\theta) - z(\theta)| \\ &= \left\| x_0 + \int_0^\theta u(t)dt - x_0 - \int_0^\theta w(t)dt \right\| \\ &= \left\| \int_0^\theta u(t)dt - \int_0^\theta w(t)dt \right\| \\ &= \left\| \int_0^\theta \frac{\rho}{\bar{\rho}} w(t)dt - \int_0^\theta w(t)dt \right\| \\ &= \left| \frac{\rho}{\bar{\rho}} - 1 \right| \left\| \int_0^\theta w(t)dt \right\| \\ &\leq \left| \frac{l}{\bar{\rho}\theta} \right| \left(\int_0^\theta \|w\|^2 dt \right)^{\frac{1}{2}} \\ &\leq \frac{l}{\bar{\rho}\theta} \theta (\bar{\rho}^2)^{\frac{1}{2}} \end{aligned}$$

Hence, the strategy constructed ensures completion of pursuit in the game G_2 and G_4 in the l-catch sense. The admissibility of this strategy is from the fact on the geometric constraints that

$$\|u(t)\| = \left\| \frac{\rho}{\bar{\rho}} w(t) \right\| = \left| \frac{\rho}{\bar{\rho}} \right| \|w(t)\| \leq \rho$$

Example 1 consider the differential game described by:

$$\begin{cases} P : \dot{x}(t) = u(t), & x(0) = x_0, \\ E : \dot{y}(t) = v(t), & y(0) = y_0, \end{cases}$$

Let $l = 0.5$ then $\bar{\rho} = 5 + \frac{0.5}{2} = 5.25$, $x_0 = (0,0,0, \dots)$ and $y_0 = (0,0,2,0,0, \dots)$, the admissible control u and v are such that

$$\int_0^4 \|u(t)\|^2 dt \leq 25, \quad \int_0^4 \|v(t)\|^2 dt \leq 16$$

The dummy pursuer \bar{P} will use the control $w(\cdot)$ and the following restriction will be impacted on it.

$$\int_0^4 ||w(t)||^2 dt \leq 27.5625,$$

Proof We show the admissibility of the strategy $u(t) = \frac{\rho}{\bar{\rho}} w(t)$ as follows:

$$\begin{aligned} \int_0^4 ||u(t)||^2 dt &= \int_0^4 \left| \left| \frac{5}{5.25} w(t) \right| \right|^2 dt \\ &= \left| \frac{5}{5.25} \right|^2 \int_0^4 ||w(t)||^2 dt \\ &\leq \left(\frac{5}{5.25} \right)^2 \times 27.5625 = 25 \end{aligned}$$

Moreover,

$$\begin{aligned} ||x(4) - y(4)|| &= ||x(4) - Z(4)|| \\ &= ||0 + \int_0^4 u(t) dt - 0 - \int_0^4 w(t) dt|| \\ &= || \int_0^4 \frac{5}{5.25} w(t) dt - \int_0^4 w(t) dt || \\ &= \left| \frac{5}{5.25} - 1 \right| \left| \int_0^4 w(t) dt \right| \\ &\leq 0.0476 \left(\int_0^4 dt \right)^{\frac{1}{2}} \left(\int_0^4 ||w||^2 dt \right)^{\frac{1}{2}} \\ &\leq 0.0476 \times \sqrt{4} \times (5.25^2)^{\frac{1}{2}} \\ &= 0.5 \end{aligned}$$

Therefore, by Theorem 1, the completion of pursuit in the game is possible in l-catch sense

Example 2 consider the differential game described by:

$$\begin{cases} P : \dot{x}(t) = u(t), & x(0) = x_0, \\ E : \dot{y}(t) = v(t), & y(0) = y_0, \end{cases}$$

Let $l = 0.05$ then $\bar{\rho} = 6 + \frac{0.05}{8} = 5.25$, $x_0 = (0,0,0, \dots)$ and $y_0 = (0,0,5,0,0, \dots)$, the admissible control u and v are such that:

$$||u(t)|| \leq 6, \quad t \in [0,8] \quad ||v(t)|| \leq 4, \quad t \in [0,8],$$

The dummy pursuer \bar{P} will use the control $w(\cdot)$ and the following restriction will be applied on it.

$$||w(t)|| \leq 6.00625$$

Proof We show the admissibility of the strategy $u(t) = \frac{\rho}{\bar{\rho}} w(t)$ as follows:

$$||u(t)|| = \left| \left| \frac{6}{6.00625} w(t) \right| \right| = \left| \frac{6}{6.00625} \right| ||w(t)|| \leq 6$$

Moreover,

$$||x(8) - y(8)|| = ||x(8) - Z(8)||$$

$$\begin{aligned}
 &= \left\| 0 + \int_0^8 u(t) dt - 0 - \int_0^8 w(t) dt \right\| \\
 &= \left\| \int_0^8 \frac{6}{6.00625} w(t) dt - \int_0^8 w(t) dt \right\| \\
 &= \left| \frac{6}{6.00625} - 1 \right| \left\| \int_0^8 w(t) dt \right\| \\
 &\leq \frac{0.00625}{6.00625} \times 6.00625 \times 8 \\
 &= 0.05
 \end{aligned}$$

Therefore, by Theorem 2, the completion of pursuit in the game is possible in l-catch sense

5. Conclusions

In this research work, we have proposed a fixed duration simple pursuit differential game problem of l catch of a single pursuer and evader in the Hilbert space l_2 . The control functions of the players are either subjected to integral or geometric constraints. Two theorems are formulated, the first is applied when the pursuer uses integral constraints, while the second theorem can be applied when the pursuer uses the geometric constraints. Two illustrative examples are given to support the theorems. Our proposed results complement the work of Badakaya [3] from \mathbb{R}^n to l_2 spaces.

Competing interests.

The authors declare that they have no conflict of interest. Authors' contributions. In writing this article, YAH formulated the problems and drafted the first version of the manuscript. SS, IA and SM reviewed and organized the results to the present form. All authors read and approved the final manuscript.

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