



Economic Order Quantity Model for Delayed Deteriorating Items with Time Dependent Linear Declining Demand and Backordering

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Abstract

In this paper, an Economic Order Quantity (EOQ) Inventory model was developed for delayed (Non-instantaneous) deteriorating items with the assumption that the demand rate before deterioration begins is linear declining function of time while the demand rate when deterioration sets in becomes constant up to when the inventory is completely depleted. The optimal cycle length that gives the minimum total inventory cost was at the end determined and the maximum backorder level determined. The model was solved analytically by minimizing the total inventory cost which leads to the determination of the best cycle length. Numerical examples were provided to illustrate the application of the developed model

Keywords: *EOQ, Inventory, Delayed deterioration, linear declining demand.*

1. Introduction

Inventory can be defined as the stock of goods, commodities and other economic resources that are stored or reserved in order to ensure smooth and efficient running of businesses. Inventory may include stationeries, petroleum products, raw materials, furniture, machinery, blood in blood banks and so on. The purpose of holding inventories in a manufacturing firm is to facilitate a constant and smooth rate of production for the firm which may include holding stocks of raw materials and semi-finished goods so that the production is not interrupted even if the industrial units supplying the materials are frequently troubled by power shortages, transport bottlenecks, labor unrest and so on.

On the other hand, a company holds stocks of finished goods so that it can meet excess demand, out of the inventories accumulated during period of low demand. A retail outlet holds stock so as to meet its usual day-to-day demand while inventories such as furniture, electronics, stationeries, drugs are held so as to meet the social, physical, educational and other needs of the society. The study of inventory model comes into force in 1915. Harris [1] was the first mathematician who carried out on inventory problems. He developed the simple but famous EOQ formula that was also derived, apparently independently, by Wilson [2]. Gradually, demand of goods may vary with time or with price or with the instantaneous inventory level displayed in a market. In recent years, inventory modelers are working for finding the economic replenishment policy for an inventory system having time dependent demand pattern. Silver and Meal [3] first developed a heuristic approach to determine EOQ in the general case of a time varying-demand pattern. Donaldson [4] first constructed a model on come out with a full analytic solution of the inventory replenishment problem with a linear trend in demand pattern over a finite-time horizon. Musa and Sani [5] constructed an inventory model of delayed deteriorating items under permissible delay in payment. Musa and Sani [6] developed an EOQ model for delayed deteriorating items with linear time dependent holding cost. Khanra, etj, al. [7] developed an inventory model considering time-quadratic demand rate. During a delay period (or trade credit period) suppliers usually offer their retailers a certain credit period with interest during the permissible delay period. Goyal [8] first developed the EOQ model under the conditions of permissible delay in payment. Yahaya, and Abubakar. [9] Developed an Economic Order Quantity

for Delayed Deteriorating items with Time Dependent Exponential Declining Demand. Adamu and Musa. [10] Determination of the Economic Order Quantity for Delayed Deteriorating Items with Linear Trended Demand and Partial Backlogging. Adamu and Musa. [11] Developed an EOQ Model for Delayed Deteriorating Items with Quadratic Time Dependent Holding Cost and Backordering. Abubakar and Adamu. [12] Production-Inventory Model of Delayed Deteriorating Items with Quadratic and Price Dependent Demands and Backordering. In this paper an inventory model for delayed deteriorating items with linear time dependent holding cost and backordering is developed. The retailer in this situation does not allow for backordering. The items backordered are settled first when a new replenishment account is received.

2. Related Works

The demand rate is deterministic and is an linear declining function of time before deterioration sets in, i.e. in the interval $0 \leq t \leq T_1$ The demand rate is constant when deterioration sets in the interval, $T_1 \leq t \leq T$ Deteriorating rate is considered as constant, $\theta(t) = \theta$, $0 < \theta < 1$ Shortages are not allowed Backordering allowed Replenishment is instantaneous Lead time is zero

2.1 Notations

$K_1(t)$ The Linear demand rate (units per unit time) before deterioration set in,

$$K_1(t) = \alpha_1 + \alpha_2 t$$

K_2 The demand rate (units per unit time) when the deterioration sets in

T_1 The time when the deterioration begins

T_2 The period of deterioration

T The inventory cycle length that is time interval between two

C The unit cost of the item (Naira per unit)

i Inventory carrying charge per unit time (Naira per unit per unit time)

A The fixed ordering or replenishment cost (Naira per order)

I_0 The initial inventory

B_c The backorder cost per unit time

$I(t)$ The inventory level at any time t before deterioration begins

I_d The inventory level at the time deterioration begins

$I_d(t)$ The inventory level at any time t after deterioration sets in

T_d Total demand over the period of deterioration $[T_1, T]$

$TC(T)$ Total inventory cost per unit time (Naira per unit time)

Mathematical Formulation and Solution

Diagrammatic representation of the model

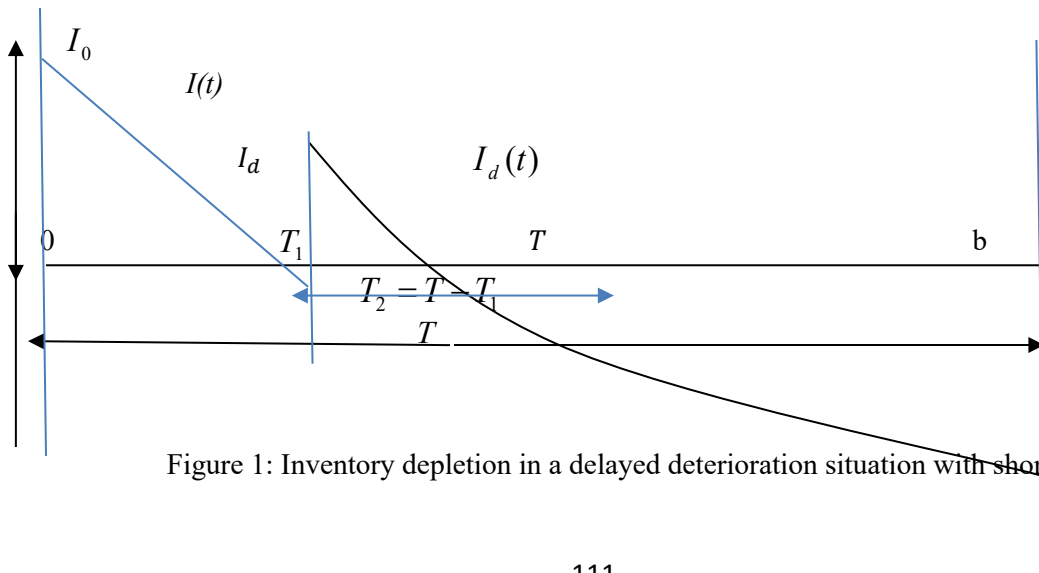


Figure 1: Inventory depletion in a delayed deterioration situation with shortages

The inventory status before the deterioration sets in is governed by the differential equation:

$$\frac{dI(t)}{dt} = -K_1(t) \quad 0 \leq t \leq T_1 \quad (1)$$

The differential equation representing the status of the inventory when the deterioration sets in is given by:

$$\frac{dI(t)}{dt} + \theta I_d(t) = -K_2(t) \quad T_1 \leq t \leq T \quad (2)$$

Equation (1) is first order variable separable, thus,

$$\frac{dI(t)}{dt} = -(\alpha_1 + \alpha_2 t) \quad \text{where } K_1(t) = (\alpha_1 + \alpha_2 t) \quad \text{where}$$

$$I(t) = - \int (\alpha_1 + \alpha_2 t) dt$$

$$I(t) = -\alpha_1 t - \frac{\alpha_2 t^2}{2} + C \quad (3)$$

Substituting the boundary condition, at $t = 0$, $I(t) = I_0$ in equation (3), we get $I_0 = C$ so that

$$I(t) = -\alpha_1 t - \frac{\alpha_2 t^2}{2} + I_0$$

$$I(t) = I_0 - \alpha_1 t - \frac{\alpha_2 t^2}{2} \quad (4)$$

Putting the condition $t = T_1$, $I(t) = I_d$ in equation (4) we get,

$$I_d = I_0 - \alpha_1 T_1 - \frac{\alpha_2 T_1^2}{2}$$

$$I_0 = I_d + T_1 \left(\alpha_1 + \frac{\alpha_2 T_1}{2} \right) \quad (5)$$

Substituting (5) in (4) we get,

$$I(t) = I_d + T_1 \left(\alpha_1 + \frac{\alpha_2 T_1}{2} \right) - \alpha_1 t - \frac{\alpha_2 t^2}{2}$$

$$I(t) = I_d + T_1 \alpha_1 - \alpha_1 t + \frac{\alpha_2 T_1^2}{2} - \frac{\alpha_2 t^2}{2}$$

$$I(t) = I_d + \alpha_1 (T_1 - t) + \frac{\alpha_2}{2} (T_1^2 - t^2) \quad (6)$$

Equation (2) is first order linear differential equation with integrating factor $I.F = e^{\int \theta dt} = e^{\theta t}$. The solution of (2) is thus given by:

$$I.F I_d(t) = \int I.F Q(t) dt. \quad \text{where } P(t) = \theta \text{ and } Q(t) = -K_2$$

$$I_d(t) e^{\theta t} = -K_2 \int e^{\theta t} dt$$

$$= I_d(t) e^{\theta t} = \frac{-K_2 e^{\theta t}}{\theta} + C_1$$

Where C_1 is arbitrary constant

$$I_d(t) = \frac{-K_2}{\theta} + C_1 e^{-\theta t} \quad (7)$$

Putting the condition, at $t = T_1$, $I_d(t) = I_d$ in equation (7) we get,

$$I_d = \frac{-K_2}{\theta} + C_1 e^{-\theta T_1}$$

$$= C_1 e^{-\theta T_1} = I_d + \frac{K_2}{\theta}$$

$$C_1 = \left(I_d + \frac{K_2}{\theta} \right) e^{\theta T_1} \quad (8)$$

Putting equation (8) in (7) we get,

$$\begin{aligned}
 I_d(t) &= \frac{-K_2}{\theta} + \left(I_d + \frac{K_2}{\theta} \right) e^{\theta T_1} \cdot e^{-\theta t} \\
 &= \frac{-K_2}{\theta} + \left(I_d + \frac{K_2}{\theta} \right) e^{\theta(T_1-t)} \\
 &= \frac{-K_2}{\theta} + I_d e^{\theta(T_1-t)} + \frac{K_2}{\theta} e^{\theta(T_1-t)} \\
 I_d(t) &= I_d e^{\theta(T_1-t)} + \frac{K_2}{\theta} (e^{\theta(T_1-t)} - 1) \tag{9}
 \end{aligned}$$

Putting the condition at $t = T$, $I_d(t) = 0$ in equation (9) we get,

$$\begin{aligned}
 0 &= I_d e^{\theta(T_1-T)} + \frac{K_2}{\theta} (e^{\theta(T_1-T)} - 1) \\
 I_d e^{\theta(T_1-T)} &= -\frac{K_2}{\theta} (e^{\theta(T_1-T)} - 1) \\
 I_d &= -\frac{K_2}{\theta} (e^{\theta(T_1-T)} - 1) e^{-\theta(T_1-T)} \\
 &= -\frac{K_2}{\theta} + \frac{K_2}{\theta} e^{-\theta(T_1-T)} \\
 I_d &= \frac{K_2}{\theta} (e^{-\theta(T_1-T)} - 1) \\
 I_d &= -\frac{K_2}{\theta} (1 - e^{\theta(T-T_1)}) \tag{10}
 \end{aligned}$$

Substituting equation (10) in to equation (6) we get,

$$I(t) = -\frac{K_2}{\theta} (1 - e^{\theta(T-T_1)}) + \alpha_1(T_1 - t) + \frac{\alpha_2}{2} (T_1^2 - t^2) \tag{11}$$

Again substituting (10) into equation (9) we get,

$$\begin{aligned}
 I_d(t) &= -\frac{K_2}{\theta} (1 - e^{\theta(T-T_1)}) e^{\theta(T_1-t)} + \frac{K_2}{\theta} (e^{\theta(T_1-t)} - 1) \\
 I_d(t) &= -\frac{K_2}{\theta} e^{\theta(T_1-t)} + \frac{K_2}{\theta} e^{\theta(T-T_1)} \cdot e^{\theta(T_1-t)} + \frac{K_2}{\theta} e^{\theta(T_1-t)} - \frac{K_2}{\theta} \\
 I_d(t) &= -\frac{K_2}{\theta} e^{\theta(T_1-t)} + \frac{K_2}{\theta} e^{\theta(T-t)} + \frac{K_2}{\theta} e^{\theta(T_1-t)} - \frac{K_2}{\theta} \\
 I_d(t) &= \frac{K_2}{\theta} (e^{\theta(T-t)} - 1) \tag{12}
 \end{aligned}$$

The total demand over the period of deterioration $[T_1, T]$ is $T_d = K_2 \times (T - T_1) = K_2 T_2$

The number of items in the inventory that deteriorated

$$N_d = I_d - K_2 T_2 \tag{13}$$

Substituting for I_d from equation (10) into equation (13) we get,

$$\begin{aligned}
 N_d &= -\frac{K_2}{\theta} (1 - e^{\theta(T-T_1)}) - K_2 T_2 \\
 &= -\frac{K_2}{\theta} + \frac{K_2}{\theta} e^{\theta(T-T_1)} - K_2 T_2 \\
 &\quad - \frac{K_2}{\theta} [1 - e^{\theta(T-T_1)} + \theta T_2] \\
 N_d &= -\frac{K_2}{\theta} [1 - e^{\theta(T-T_1)} + (T - T_1)\theta] \tag{14}
 \end{aligned}$$

Computation of variable cost

The total inventory (variable) cost is made up of the sum of the following inventory related costs:

- i. Ordering Cost, $O_c = A$
- ii. Inventory holding (carrying) cost, H_c
- iii. Cost of deteriorated items CN_d

iv. Back order cost per cycle B_C

These costs are calculated separately and then sum up to get the total variable cost.

The inventory holding (carrying) cost, H_c

$$\begin{aligned}
 H_c &= ic \int_0^{T_1} I(t)dt + ic \int_{T_1}^T I_d(t)dt \\
 &= ic \int_0^{T_1} -\frac{K_2}{\theta}(1 - e^{\theta(T-T_1)}) + \alpha_1(T_1 - t) + \frac{\alpha_2}{2}(T_1^2 - t^2) dt + ic \int_{T_1}^T \frac{K_2}{\theta}(e^{\theta(T-t)} - 1)dt \\
 &= ic \int_0^{T_1} \left(-\frac{K_2}{\theta} + \frac{K_2 e^{\theta(T-T_1)}}{\theta} + \alpha_1 T_1 - \alpha_1 t + \frac{\alpha_2 T_1^2}{2} - \frac{\alpha_2 t^2}{2} \right) dt + ic \int_{T_1}^T \left(\frac{K_2 e^{\theta(T-t)}}{\theta} - \frac{K_2}{\theta} \right) dt \\
 &\quad - \frac{icK_2}{\theta} \int_0^{T_1} dt + \frac{icK_2 e^{\theta(T-T_1)}}{\theta} \int_0^{T_1} dt + ic\alpha_1 T_1 \int_0^{T_1} dt - ic\alpha_1 \int_0^{T_1} t dt + \frac{ic\alpha_2 T_1^2}{2} \int_0^{T_1} dt - \frac{ic\alpha_2}{2} \int_0^{T_1} t^2 dt \\
 &\quad + \frac{icK_2}{\theta} \int_{T_1}^T e^{\theta(T-t)} dt - \frac{icK_2}{\theta} \int_{T_1}^T dt \\
 &\quad - \frac{icK_2 T_1}{\theta} + \frac{icK_2 T_1 e^{\theta(T-T_1)}}{\theta} + ic\alpha_1 T_1^2 - \frac{ic\alpha_1 T_1^2}{2} + \frac{ic\alpha_1 T_1^3}{2} - \frac{ic\alpha_2 T_1^3}{6} - \frac{icK_2}{\theta^2} + \frac{icK_2 e^{\theta(T-T_1)}}{\theta^2} \\
 &\quad - \frac{icK_2 T}{\theta} + \frac{icK_2 T_1}{\theta} \\
 &= \frac{icK_2 T_1}{\theta} \left\{ e^{\theta(T-T_1)} + \frac{\theta\alpha_1 T_1}{K_2} - \frac{\theta\alpha_1 T_1}{2K_2} + \frac{\theta\alpha_1 T_1^2}{2K_2} - \frac{\alpha_2 T_1^2}{6K_2} - \frac{1}{\theta T_1} + \frac{e^{\theta(T-T_1)}}{\theta T_1} - \frac{T}{T_1} \right\} \quad (15)
 \end{aligned}$$

Cost of deteriorated items

$$\text{Cost of deteriorated items, } CN_d = C \left[-\frac{K_2}{\theta} [1 - e^{\theta(T-T_1)} + (T - T_1)\theta] \right] \quad (16)$$

Where C is a unit cost of item.

Back order cost per cycle B_C is given as

$$B_C = C_b \int_0^{T-T_2} K_2 t dt$$

$$\frac{C_b K_2 (T - T_2)^2}{2}$$

The total variable cost per unit time is given by

$TC(T) = \frac{1}{T} \{ \text{Inventory ordering cost, } O_c + \text{inventory holding, } H_c + \text{cost of deteriorated items, } CN_d + \text{back order Cost per cycle, } B_C \}$

$$\frac{1}{T} \{ O_c + H_c + CN_d + B_C \}$$

$$\frac{A}{T} + \frac{icK_2 T_1}{T\theta} \left\{ e^{\theta(T-T_1)} + \frac{\theta\alpha_1 T_1}{K_2} - \frac{\theta\alpha_1 T_1}{2K_2} + \frac{\theta\alpha_1 T_1^2}{2K_2} - \frac{\alpha_2 T_1^2}{6K_2} + \frac{1}{\theta T_1} - \frac{e^{\theta(T-T_1)}}{\theta T_1} - \frac{T}{T_1} \right\}$$

$$+ \frac{A}{T} \left[-\frac{K_2}{\theta} [1 - e^{\theta(T-T_1)} + (T - T_1)\theta] \right] + \frac{C_b K_2 (T - T_2)^2}{2T} \quad (17)$$

Our aim is to optimize the variable cost per unit time. The necessary and sufficient conditions that minimize our

$TC(T)$ for a given value of T are respectively: $\frac{dTC(T)}{dT} = 0$ and $\frac{d^2TC(T)}{dT^2} > 0$.



$$\frac{dT(T)}{dT} = -\frac{A}{T^2} + \frac{icK_2 T_1}{\theta T^2} \left\{ (T - 1)e^{\theta(T-T_1)} - \frac{\theta\alpha_1 T_1}{2K_2} + \frac{T_1^2}{2K_2} \left(\frac{\alpha_2}{3} - \theta\alpha_1 \right) - \frac{1}{\theta T_1} - \frac{1}{\theta T_1} \left(\frac{T}{\theta} - 1 \right) e^{\theta(T-T_1)} \right\} + \frac{A}{T^2} \left[-\frac{K_2}{\theta} \left(-1 + \left(\frac{T}{\theta} - 1 \right) e^{\theta(T-T_1)} + \theta - (T - T_1)\theta \right) \right] + \frac{C_b K_2}{2T^2} (2T(T - T_2) - (T - T_2)^2) = 0 \quad (18)$$

Multiply equation (18) by T^2 and equating to zero

$$-A + \frac{icK_2 T_1}{\theta} \left\{ (T - 1)e^{\theta(T-T_1)} - \frac{\theta\alpha_1 T_1}{2K_2} + \frac{T_1^2}{2K_2} \left(\frac{\alpha_2}{3} - \theta\alpha_1 \right) - \frac{1}{\theta T_1} - \frac{1}{\theta T_1} \left(\frac{T}{\theta} - 1 \right) e^{\theta(T-T_1)} \right\} + A \left[-\frac{K_2}{\theta} \left(-1 + \left(\frac{T}{\theta} - 1 \right) e^{\theta(T-T_1)} + \theta - (T - T_1)\theta \right) \right] + \frac{C_b K_2}{2} (2T(T - T_2) - (T - T_2)^2) = 0 \quad (19)$$

We use (19) to determine the best cycle length T which minimize the total inventory cost. The economic order quantity corresponding to the optimal cycle length $T = T^*$ can be computed as follows: EOQ = Total demand in the period $[0, T_1]$ before deteriorated sets in + Total demand over the deterioration period $[T_1, T]$ + Number of items that deteriorated

$$K_1(t)T_1 + K_2T_2 + N_d$$

$$(\alpha_1 + \alpha_2 t)T_1 + K_2(T - T_1) - \frac{K_2}{\theta} [1 - e^{\theta(T-T_1)} + (T - T_1)\theta]$$

$$(\alpha_1 + \alpha_2 t)T_1 + \frac{K_2}{\theta} [e^{\theta(T-T_1)} - 1] \quad (20)$$

3.0 Methodology

3.1 Numerical Examples

The solutions of five different numerical examples representing the application of the model are given in the table below

4.0 Table for Numerical Examples

| S/N | A (Naira) | C | K_1 (Units) | K_2 (Units) | i | T_1 | T_2 | θ | α_1 | α_2 | T | C_b | $TI_c(T)$ | EOQ (Units) |
|-----|-----------|----|---------------|---------------|-------|--------|--------|----------|------------|------------|--------|-------|-----------|-------------|
| 1 | 50 | 20 | 100 | 150 | 0.013 | 0.0162 | 0.0435 | 0.1 | 250 | 200 | 0.4421 | 20 | 10927 | 360 |
| 2 | 60 | 30 | 150 | 200 | 0.014 | 0.0520 | 0.0432 | 0.2 | 600 | 500 | 0.7610 | 15 | 6859 | 512 |
| 3 | 70 | 40 | 200 | 300 | 0.015 | 0.0620 | 0.0532 | 0.3 | 700 | 600 | 0.8063 | 10 | 11170 | 579 |
| 4 | 80 | 50 | 250 | 400 | 0.016 | 0.0720 | 0.0632 | 0.4 | 800 | 700 | 0.6469 | 05 | 31322 | 700 |
| 5 | 90 | 60 | 300 | 500 | 0.017 | 0.0820 | 0.0732 | 0.5 | 900 | 800 | 0.7434 | 05 | 32593 | 944 |

5.0 Conclusion

In this study, we developed an inventory model that is applicable to a business environment that considers the fact that deterioration of the storage item is delayed (non-instantaneous) and in which the demand rate before deterioration begins is linear declining function of time and when deterioration sets in, it is constant up to the end of the cycle. The model does not allowed for shortages. The model was solved analytically to determine the minimum total inventory cost, the optimal cycle length and the Economic Order Quantity (EOQ). Five different parameter values are taken to provide numerical examples on the application of the model. The result obtained from the numerical examples indicate the validity and stability of the model.

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