

# Jaggi-Type Contraction Mapping in $(\alpha, \beta)$ - Complex Valued b-Metric Space

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## Abstract

*In this paper, we established the existence of fixed point for Jaggi-type contraction in  $(\alpha, \beta)$  - complex valued b-metric space. The proved results extend and generalized some results in the literature.*

**Keywords:** *fixed point, contraction, complex valued b-metric space.*

## 1. Introduction

Fixed point theory is self-sufficient of the cutting-edge topics in both Pure and Applied Mathematics. It has seen abundant interest for several decades, because it is considered an important tool for Nonlinear Analysis and many other branches of contemporary Mathematics. Fixed point theory is one of the prominent and outmoded theories in Mathematics and has a diverse set of applications. In this theory contraction is one of the foremost tools to prove the existence and uniqueness of a fixed point.

## 2. Related Works

In 1989, Bakhtin introduced the idea of a b-metric space as a generalization of metric space, which was used in 1993 by Czerwik and presented by an axiom which was punier than the triangular inequality and properly defined a b-metric space with an outlook of oversimplifying the Banach contraction mapping principle.

In 2011, Azam and co-authors introduced the idea of complex valued metric space which is a generality of the conventional metric space and established the existence of common fixed point theorems for mappings satisfying some contraction-type circumstances.

In 2013, Rao and co-workers introduced the idea of complex valued b-metric space and prove some fixed point results of different contractions.

In 2021, Singh and co-authors introduced the idea of  $(\alpha, \beta)$  – complex valued b-metric space and prove some fixed point results of different contractions in such a space.

Let  $\mathbb{C}$  be the set of complex numbers. Suppose  $k_1, k_2 \in \mathbb{C}$  and define a partial order  $\leq$  on  $\mathbb{C}$  as follows:

$$k_1 \leq k_2 \text{ if and only if } \operatorname{Re}(k_1) \leq \operatorname{Re}(k_2), \operatorname{Im}(k_1) \leq \operatorname{Im}(k_2).$$

It follows that  $k_1 \leq k_2$  if and only if one of the following circumstances is satisfied:

- (I)  $\text{Re}(k_1) = \text{Re}(k_2), \text{Im}(k_1) < \text{Im}(k_2),$
- (II)  $\text{Re}(k_1) < \text{Re}(k_2), \text{Im}(k_1) = \text{Im}(k_2),$
- (III)  $\text{Re}(k_1) < \text{Re}(k_2), \text{Im}(k_1) < \text{Im}(k_2),$
- (IV)  $\text{Re}(k_1) = \text{Re}(k_2), \text{Im}(k_1) = \text{Im}(k_2).$

Specifically, we will write  $k_1 \preceq k_2$  if  $k_1 = k_2$  and one of (i), (ii), and (iii) is satisfied and we will write  $k_1 < k_2$  if only (iii) is satisfied. Kindly observe

$$0 \leq k_1 \preceq k_2 \text{ implies that } |k_1| < |k_2|,$$

$$k_1 \leq k_2, k_2 \leq k_3 \text{ implies that } k_1 \leq k_3,$$

$$\text{If } x, y \in \mathbb{R} \text{ and } x \leq y \text{ then } xk \leq yk \text{ for all } k \in \mathbb{C}.$$

**Definition One.** (Azam et al., 2011) Let  $A$  be a nonempty set. Suppose that the mapping  $\rho : A \times A \rightarrow \mathbb{C}$  satisfies:

- (I)  $0 \leq \rho(a, b), \forall a, b \in A$  and  $\rho(a, b) = 0 \iff a = b;$
- (II)  $\rho(a, b) = \rho(b, a), \forall a, b \in A;$
- (III)  $\rho(a, b) \leq \rho(a, c) + \rho(c, b), \forall a, b, c \in A.$

Then  $\rho$  is called a complex-valued metric on  $A$ , and  $(A, \rho)$  is called a complex-valued metric space.

**Definition Two.** (Rao et al., 2013) Let  $A$  be a nonempty set and  $m \geq 1$  be a given real number. Suppose that the mapping  $\rho : A \times A \rightarrow \mathbb{C}$ , satisfies:

- (I)  $0 \leq \rho(a, b), \forall a, b \in A$  and  $\rho(a, b) = 0 \iff a = b;$
- (II)  $\rho(a, b) = \rho(b, a), \forall a, b \in A;$
- (III)  $\rho(a, b) \leq m[\rho(a, c) + \rho(c, b)], \forall a, b, c \in A.$

Then  $\rho$  is called a complex-valued b-metric on  $A$ , and  $(A, \rho)$  is called a complex-valued b-metric space.

**Example One.** Let  $A = \mathbb{C}$ . Define the mapping  $\rho : A \times A \rightarrow \mathbb{C}$  for all  $k_1, k_2 \in X$ , by

$$\rho(k_1, k_2) = |a_1 - a_2|^2 + i|b_1 - b_2|^2, \text{ where } k_1 = a_1 + ib_1 \text{ and } k_2 = a_2 + ib_2.$$

Clearly, the pair  $(A, \rho)$  is a complex valued b-metric space with  $m = 2$ .

**Definition Three.** (Singh et al., 2021) Let  $A$  be a nonempty set and  $\alpha, \beta \geq 1$  be given real numbers. Suppose that the mapping  $\rho : A \times A \rightarrow \mathbb{C}$ , satisfies:

- (I)  $0 \leq \rho(a, b), \forall a, b \in A$  and  $\rho(a, b) = 0 \iff a = b;$
- (II)  $\rho(a, b) = \rho(b, a), \forall a, b \in A;$
- (III)  $\rho(a, b) \leq \alpha \rho(a, c) + \beta \rho(c, b), \forall a, b, c \in A..$

Then  $\rho$  is called an  $(\alpha, \beta)$  – complex valued b-metric on  $A$ , and  $(A, \rho)$  is called an  $(\alpha, \beta)$  - complex valued b-metric space.

**Example Two.** Let  $A = \{1,3\}$ . Define the mapping  $\rho : A \times A \rightarrow \mathbb{C}, \forall a, b \in A$ , by

$$\rho(a, b) = e^{i|a-b| + i|a-b|}, \text{ if } a \neq b \text{ and } \rho(a, b) = 0 \iff a = b.$$

Clearly, the pair  $(A, \rho)$  is an  $(\alpha, \beta)$  complex valued b-metric space.

**Definition Four.** (Singh et al., 2021) Let  $(A, \rho)$  be an  $(\alpha, \beta)$  - complex valued b-metric space, and let  $\{x_n\}$  be a sequence in  $A$  and  $x \in A$ . Then:

- (I) The sequence  $\{x_n\}$  converges to  $x \in A$ , if for every  $0 < c \in \mathbb{C}$  then there is  $N \in \mathbb{N}$  such that  $\rho(x_n, x) < c$ . The sequence  $\{x_n\}$  converges to  $x \in A \iff |\rho(x_n, x)| \rightarrow 0$  as  $n \rightarrow \infty$ .
- (II) The sequence  $\{x_n\}$  is a Cauchy in  $(A, \rho)$  if for every  $c \in \mathbb{C}$  there is  $N \in \mathbb{N}$  such that  $\rho(x_n, x_{n+m}) < c$ , where  $m \in \mathbb{N}$ . The sequence  $\{x_n\}$  is a Cauchy in  $(A, \rho) \iff |\rho(x_n, x_{n+m})| \rightarrow 0$  as  $n \rightarrow \infty$ , where  $m \in \mathbb{N}$ .
- (III) The space  $(A, \rho)$  is complete if every Cauchy sequence  $\{x_n\}$  in  $A$  converges to a point  $x \in A$ .

### 3. Main Results

In this section, we state and prove a Jaggi -type contraction mapping in  $(\alpha, \beta)$  - complex valued b-metric space.

**Theorem:** Let  $(X, P)$  be a complete  $(\alpha, \beta)$ -complex valued b-metric space. Suppose that a mapping  $F: X \rightarrow X$  satisfies the rational condition.

$$P(Fx, Fy) \leq \lambda_1 P(x, y) + \lambda_2 \left[ \frac{P(x, Fx)P(y, Fy)}{1 + P(x, y)} \right] \tag{3.1}$$

for all  $x, y \in X$ , where  $\lambda_1, \lambda_2$  are non-negative real numbers such that  $\lambda_1 + \lambda_2 < 1$ . Then  $F$  has a unique fixed point in  $X$ .

**Proof:** Let  $x_0 \in X$  be arbitrary chosen and define an iterative sequence  $\{x_n\}$  of the form:

$$x_{n+1} = Fx_n, \text{ then}$$

$$P(x_n, x_{n+1}) = P(Fx_{n-1}, Fx_n).$$

By using contractive condition of Jaggi define as:

$$P(Fx, Fy) \leq \lambda_1 P(x, y) + \lambda_2 \left[ \frac{P(x, Fx)P(y, Fy)}{1 + P(x, y)} \right].$$

Put  $x = x_{n-1}$  and  $y = x_n$ ; by substituting the value of  $x$  and  $y$  in above contractive condition, we have:

$$\begin{aligned} P(x_n, x_{n+1}) &= P(Fx_{n-1}, Fx_n) \\ &\leq \lambda_1 P(x_{n-1}, x_n) + \lambda_2 \left[ \frac{P(x_{n-1}, Fx_{n-1})P(x_n, Fx_n)}{1 + P(x_{n-1}, x_n)} \right] \end{aligned} \tag{3.2}$$

Recall that  $x_n = Fx_{n-1}$  and also  $Fx_n = x_{n+1}$ , by substituting  $Fx_{n-1} = x_n$ , and also  $Fx_n = x_{n+1}$  in the contractive condition above, we have:

$$P(x_n, x_{n+1}) \leq \lambda_1 P(x_{n-1}, x_n) + \lambda_2 \left\{ \frac{P(x_{n-1}, x_n)P(x_n, x_{n+1})}{1 + P(x_{n-1}, x_n)} \right\}. \tag{3.3}$$

Since  $P(x_{n-1}, x_n) \leq 1 + P(x_{n-1}, x_n)$ ,

$$\begin{aligned}
 P(x_n, x_{n+1}) &\leq \lambda_1 P(x_{n-1}, x_n) + \lambda_2 P(x_n, x_{n+1}), \\
 [P(x_n, x_{n+1}) - \lambda_2 P(x_n, x_{n+1})] &\leq \lambda_1 P(x_{n-1}, x_n), \\
 (1 - \lambda_2)P(x_n, x_{n+1}) &\leq \lambda_1 P(x_{n-1}, x_n), \\
 P(x_n, x_{n+1}) &\leq \frac{\lambda_1}{(1 - \lambda_2)} P(x_{n-1}, x_n). \tag{3.4}
 \end{aligned}$$

Let  $K = \frac{\lambda_1}{(1 - \lambda_2)}$ , then since  $\lambda_1 + \lambda_2 < 1$ , it implies that:

$$\begin{aligned}
 \lambda_1 < 1 - \lambda_2, \text{ then } \frac{\lambda_1}{(1 - \lambda_2)} < 1 \text{ that is } K < 1 \\
 P(x_n, x_{n+1}) &\leq K P(x_{n-1}, x_n) \\
 &\leq K^2 P(x_{n-2}, x_{n-1}), \\
 &\leq K^3 P(x_{n-3}, x_{n-2}), \\
 &\vdots \\
 &\leq K^n P(x_0, x_1) \tag{3.5}
 \end{aligned}$$

Next, we show that the sequence  $\{x_n\}$  is Cauchy sequence. By triangular inequality, for  $n, m$  in  $Z$ , and  $n \geq m$ ;

$$P(x_m, x_n) \leq P(x_m, x_{m+1}) + P(x_{m+1}, x_{n+2}) + P(x_{m+2}, x_{m+3}) + \dots + P(x_{n-1}, x_n) \tag{3.6}$$

By using  $P(x_n, x_{n+1}) \leq K^n P(x_0, x_1)$  in the above inequality, we have:

$$p(x_m, x_n) = K^m P(x_0, x_1) + K^{m+1} P(x_0, x_1) + K^{m+2} P(x_0, x_1) + \dots + K^{n-1} P(x_0, x_1) \tag{3.7}$$

$$P(x_m, x_n) = K^m [1 + K + K^2 + \dots + K^{n-m-1}] P(x_0, x_1) \tag{3.8}$$

$$P(x_m, x_n) = K^m \frac{(1 - K^{n-m})}{(1 - K)} P(x_0, x_1) \tag{3.9}$$

$$|P(x_m, x_n)| \leq \frac{K^m}{(1-K)} |P(x_0, x_1)| \rightarrow 0 \text{ as } n, m \rightarrow \infty \tag{3.10}$$

Since  $K < 1$ , which on taking limit as  $n, m \rightarrow \infty$  is a Cauchy sequence. Also, since  $X$  is complete there exists an element  $x^*$  in  $X$  such that  $\{x_n\}$  converges to  $x^*$ . Now we show that  $F x^* = x^*$ . If not, there exist  $z$  in  $C$  such that  $P(F x^*, x^*) = Z > 0$ . By using contractive condition for Jaggi. We have:

$$Z = P(x^*, F x^*) \tag{3.11}$$

$$Z \leq \alpha P(x^*, x_{n+1}) + \beta P(x_{n+1}, F x^*) \tag{3.12}$$

$$Z \leq \alpha P(x^*, x_{n+1}) + \beta \{ \lambda_1 P(x_n, x^*) + \lambda_2 \left[ \frac{P(x^*, F x_n) p(x^*, F x^*)}{1 + P(x_n, x^*)} \right] \} \tag{3.13}$$

$$Z \leq \alpha P(x^*, x_{n+1}) + \beta \lambda_1 P(x_n, x^*) + \beta \lambda_2 \frac{P(x_n, x_{n+1}) Z}{1 + P(x_n, x^*)} \tag{3.14}$$

This implies that:

$$|Z| \leq \alpha |P(x^*, x_{n+1})| + \beta \lambda_1 |P(x_n, x^*)| + \beta \lambda_2 \left| \frac{P(x_n, x_{n+1}) Z}{1 + P(x_n, x^*)} \right|,$$

Taking the limit as  $n \rightarrow \infty$  of both sides, we have that:  $|Z| \leq 0$ , a contradiction for that  $P(x^*, F x^*) = Z > 0$ , So  $|Z| = 0$ ,

$$P(F x^*, x^*) = 0$$

Hence  $F x^* = x^*$ . That is  $x^*$  is a fixed point of  $F$ .

**Uniqueness of the Fixed Point**

To show that the fixed point is unique, we assume  $x^*$  and  $y^*$  are two fixed point of  $F$ .

Then  $x^* = F x^*$  and  $y^* = F y^*$

$$P(x^*, y^*) = (F x^*, F y^*),$$

Applying the contractive condition for Jaggi, which is define as follows:

$$P(Fx, Fy) \leq \lambda_1 P(x, y) + \lambda_2 \left\{ \frac{P(x, Fx)P(y, Fy)}{1 + P(x, y)} \right\},$$

By substitution of the value to contractive condition,  $x^* = Fx^*$  and  $y^* = Fy^*$ , we have:

$$P(Fx^*, Fy^*) \leq \lambda_1 P(x^*, y^*) + \lambda_2 \left\{ \frac{P(x^*, Fx^*)P(y^*, Fy^*)}{1 + P(x^*, y^*)} \right\}$$

Apply the value of the form:

$$F x^* = x^* \text{ and } F y^* = y^*$$

$$P(x^*, y^*) \leq \lambda_1 P(x^*, y^*) + \lambda_2 \left\{ \frac{P(x^*, x^*)P(y^*, y^*)}{1 + P(x^*, y^*)} \right\}$$

But  $P(x^*, x^*) = 0$  and  $P(y^*, y^*) = 0$ , so we have:

$$P(x^*, y^*) < P(x^*, y^*)$$

This implies that:

$$P(x^*, y^*) = 0,$$

Thus,  $x^* = y^*$ . Hence the proved.

**4. Conclusion**

In this research work, we have seen the proof of fixed point in theorem in rational form that is Jaggi type contraction mapping in  $(\alpha, \beta)$  complex valued b-metric space. And this paper help the users, to extend the study of contractive type of mapping, and obtain sufficient conditions for existence of fixed point in complex valued b-metric space.

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