

On Absolute Irreducibility of Binomial Polynomials

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Abstract

This research is focused on investigating absolute irreducibility in $Int(\mathbb{Z})$ with special interest on binomial polynomials. We modify an existing method called the Table Method by varying some parameters to achieve our result. The findings shows that binomial polynomials are absolutely irreducible, as shown by the Table Method.

Keywords: Binomial polynomials, Absolute irreducibility, Table Method.

1. Introduction

The ring of integer valued polynomials $Int(\mathbb{Z})$ is a set of polynomials $Int(\mathbb{Z}) = \{f(x) \in \mathbb{Z}[x] \mid F(\mathbb{Z}) \subset \mathbb{Z}\}$ with rational coefficients which map the set of integers to itself. Binomial polynomials are very important class of $Int(\mathbb{Z})$ because they form a basis for $Int(\mathbb{Z})$ as \mathbb{Z} -Module.

Researchers had focused on irreducible polynomials in $Int(\mathbb{Z})$, for example, (Austine et al., 2018) presented a computational method called Table Method used to recognize all irreducible polynomial in $Int(\mathbb{Z})$, however, literatures on absolutely irreducibility in $Int(\mathbb{Z})$ are scarce. Baring the work of (Sophie and Sarah, 2020), where they gave a graph theoretical sufficient condition for absolute irreducibility of integer valued polynomials with square free denominator, absolute irreducibility in $Int(\mathbb{Z})$ had not been widely studied. More recently, by reformulating the problem of absolute irreducibility in $Int(\mathbb{Z})$ to linear algebra and number theory, (Roswitha and Daniel, 2021), showed that binomial polynomials are absolutely irreducible by reducing the question to determination of rank of valuation matrix.

However, in this research, we shall use the Table Method as an alternative method to show that binomial polynomials are absolutely irreducible. The entries in the table are used to investigate as explained in section two. In section three, some of the detailed result is presented.

2. Preliminaries

In this section, the basic preliminaries are presented.

2.1 Definition

1. Let G be an integral domain. Then all the elements in G have additive inverse but not all elements have a multiplicative inverse. Those elements with multiplicative inverse are called units.
2. A non-unit element $x \in G$ is irreducible in G if whenever $x = yz$, for all $y, z \in G$, one of y or z is a unit.

3. Two elements $x, y \in G$ are associates if there is a unit $u \in G$ such that $x = uy$.
4. G is a factorization domain if every element in G can be written as a product of irreducible elements.
5. Factorization of an element $s \in G$ is the expression $s = p_1, p_2, p_m$ where p_m are irreducible.
6. Suppose a is the factorization p_1, p_2, \dots, p_m and k is the factorization I_1, I_2, \dots, I_j , then we say a and k are said to be essentially the same if $m = j$ and after reordering, the p_1 and I_i associates. Otherwise it is essentially different.
7. An element $s \in G$ is said to absolutely irreducible if it is irreducible in G and for all natural numbers $n > 1$, every factorization of s^n is essentially the same as $s^n = s \dots s$ (n – times).
8. Let f be a polynomial, the content $c(f)$ is the ideal of \mathbb{Z} generated by the coefficients of f . The fixed divisor $d(f)$ is the ideal of \mathbb{Z} generated by the image $f(\mathbb{Z})$. Therefore, for

$$f = \sum_{k=0}^n a_k X^k \in \mathbb{Z}[X], \quad c(f) = \gcd(a_k \mid k = 0, \dots, n) \text{ and } d(f) = \gcd\{f(c) \mid c \in \mathbb{Z}\}.$$

9. A polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where a_0, a_1, \dots, a_n are integer is said to be primitive polynomial if $\gcd(a_0, a_1, \dots, a_n) = 1$.
10. A polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is image primitive if $d(f) = \mathbb{Z}$. Where $d(f) = \gcd\{f(a) \mid a \in \mathbb{Z}\}$ is the fixed divisor. but if polynomial is in this form

$$f = \frac{a \prod_{i \in I} g_i}{b} \in K[x], \tag{1}$$

for irreducibility, we require $a = 1$ and $b = d(\prod_{i \in I} g_i)$. We call the polynomial,

$$f = \frac{\prod_{i \in I} g_i}{b}, \tag{2}$$

image-primitive.

11. Let $P \subseteq \mathbb{Z}$ denote the set of (positive) prime elements and for each prime number $p \in P$, we write v_p for the p -adic valuation,

$$v_p(n) = \begin{cases} \max\{v \in \mathbb{N} : \frac{p^v}{n}\} & \text{if } n \neq 0. \\ \infty & \text{if } n = 0. \end{cases} \tag{3}$$

12. If $x, y \in \sim$. $x > y$ means that $x = y + p$, where $p \in \sim^+$. As usual, $x < y$ means $y > x$, $x \geq y$ means $x > y$ or $x = y$.
13. Let $a, b, c, d \in \sim$. such that $a > b$ and $c > d$ this implied $a + c > b + d$.

2.3. Overview of the Table Method (see, Austine et al., (2018))

The table method developed by (Austine et al., 2018) is a computational method used to recognize whether an image primitive polynomial $f \in \mathbb{Z}$ is irreducible, use p -adic valuation as a tool. We give an overview, for details see (Austine et al., 2018).

The main area of our concern in this work is the last column of the Table 1.

Table 1. Computed results up to the power of polynomial.

$a \in \mathbb{Z}$	g_1	g_2	...	$\sum_{j \in I} v_p(g_j(a))$
0	$v_p(g_1(0))$	$v_p(g_2(0))$...	$\sum_{j \in I} v_p(g_j(0))$
1	$v_p(g_1(1))$	$v_p(g_2(1))$...	$\sum_{j \in I} v_p(g_j(1))$
...	$v_p(g_1(a))$	$v_p(g_2(a))$...	$\sum_{j \in I} v_p(g_j(a))$
⋮	⋮	⋮	⋮	⋮

Remark: In our work we compute the values $\sum_{j \in J} v_p(g_j(a))$ up to the power of polynomial.

2.4. Notations

Let $G = (g_i)_{i \in I}$ be a finite family of polynomials $g_i \in \mathbb{Z}[x]$ and $J \subset I$. For a prime $p \in P$ and $a \in \mathbb{Z}$, we set

- $\sigma_G^p(I) = \min_{a \in \mathbb{Z}} \sum_{j \in J} v_p(g_j(a))$ (Austin et al. 2018).
- $\bar{\sigma}_G^p(I) = \max_{a \in \mathbb{Z}} \sum_{j \in J} v_p(g_j(a)) < \infty$.
- $\sigma_G(J) = \sum \sigma_G^p(I)$ (Austin et al. 2018).
- $\bar{\sigma}_G(I) = \sum \bar{\sigma}_G^p(I)$.
- Let $\alpha = \max\{k_p : p \in P\}$ and $\bar{\alpha} = \sum_{p \in P} \alpha$.
- Let $\beta \in \mathbb{N}^+$ be the number of prime powers of $p \in P$.
- Let $\gamma = \bar{\alpha}\beta > 0$.

In addition after compute the values of $\sum_{j \in J} v_p(g_j(a))$ we use the following Lemma and Theorem to interpret the result.

Lemma (Austin et al. 2018):

Let $G = (g_i)_{i \in I}$ be a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$ and let

$$f(x) = \frac{\prod_{i \in I} g_i(x)}{\prod_{p \in P} P^{k_p}} \in \mathbb{Z}[x]. \text{ Then,}$$

- $f \in \text{Int}(\mathbb{Z})$ if and only if $\sigma_G^p(I) \geq k_p$ for all $p \in P$ and
- $f \in \text{Int}(\mathbb{Z})$ is image-primitive if and only if $\sigma_G^p(I) = k_p$ for all $p \in P$.

Accordingly, from (4)

$$\sigma_G^p(I) \geq k_p, \forall p \in P, \tag{4}$$

this implies that.

$$\sigma_G^{\bar{P}}(I) \geq k_p \forall p \in P, \tag{5}$$

By definition (13)

$$\sum \sigma_G^{\bar{P}}(I) \geq \sum_{p \in P} k_p \forall p \in P \tag{6}$$

Theorem (Austin et al. 2018):

Let $G = (g_i)_{i \in I}$ be a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$ such that

$f(x) = \frac{\prod_{i \in I} g_i(x)}{\prod_{p \in P} P^{k_p}} \in \text{Int}(\mathbb{Z})$ is image-primitive. Then the following assertions are equivalent:

(i). f is irreducible in $\text{Int}(\mathbb{Z})$.

(ii). For all subsets $\emptyset \neq J \text{ fi } I$ there exists a prime number $p \in P$ such that

$$\sigma_G^p(J) + \sigma_G^p(I \setminus J) < k_p.$$

(iii). For all subsets $\emptyset \neq J \text{ fi } I$,

$$\sigma_G(J) + \sigma_G(I \setminus J) < \sum_{p \in P} k_p$$

2.5. Criterion for Absolute Irreducibility in Binomial Polynomials

Let $G = (g_i)_{i \in I}$ be a finite family of primitive, irreducible polynomials $g_i \in \mathbb{Z}[x]$ such that

$f(x) = \frac{\prod_{i \in I} g_i(x)}{\prod_{p \in P} P^{k_p}} \in \text{Int}(\mathbb{Z})$ is image-primitive and irreducible. If

$$\overline{\sigma_G(I)} \leq \sum_{p \in P} k_p + \gamma. \tag{7}$$

Then f is absolutely irreducible.

Proof: From Equation (2)

$$\overline{\sigma_G(I)} \geq \sum_{p \in P} k_p \tag{8}$$

Suppose

$$\overline{\sigma_G(I)} = \sum_{p \in P} k_p \quad \forall \gamma > 0, \tag{9}$$

then

$$\overline{\sigma_G(I)} < \sum_{p \in P} k_p + \gamma \tag{10}$$

Suppose

$$\overline{\sigma_G(I)} > \sum_{p \in P} k_p. \tag{11}$$

By definition (12) this is true if there exist a positive number γ such that $\overline{\sigma_G(I)} = \sum_{p \in P} k_p + \gamma$.

3. Applications

3.1. Absolute irreducible of binomial polynomials

Binomial polynomials are polynomials of the form $\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}$ for $n \geq 1$.

When $n = 1$ is straightforward, since $\binom{x}{1} = x$ is absolutely irreducible.

Example 1: For $n = 2$. We consider the polynomial,

$$f(x) = \frac{x(x-1)}{2} \tag{12}$$

We compute the Table 2 with respect to v_2 ,

Table 2. The computed results of Equation (13) with respect to v_2 .

$a \in \mathbb{Z}$	g_1	g_2	$\sum v_2$
0	∞	0	∞
1	0	∞	∞
2	1	0	1

we observed that: $\overline{\sigma_G(I)} = 1$, $\sum_{p \in P} k_p = 1$, $\bar{\alpha} = 1$, $\beta = 1$ and $\gamma = 1$. So, $1 < 1+1$. Implies that f is absolutely irreducible.

Example 2: For $n = 3$. We consider the polynomial,

$$f(x) = \frac{x(x-1)(x-2)}{6}. \tag{13}$$

We compute the Table 3 with respect to v_2 and v_3 , respectively.

Table 3. The computed results of Equation (14) with respect to v_2 and v_3 .

$a \in \mathbb{Z}$	g_1	g_2	g_3	$\sum v_2$
0	∞	0	1	∞
1	0	∞	0	∞
2	1	0	∞	∞
3	0	1	0	1
				$\sum v_3$
0	∞	0	0	∞
1	0	∞	0	∞
2	0	0	∞	∞
3	1	0	0	1

Hence, $\overline{\sigma_G(I)} = 2$, $\sum_{p \in P} k_p = 2$, $\bar{\alpha} = 2$, $\beta = 2$ and $\gamma = 4$. So, $2 < 2 + 4$. Implies that f is absolutely irreducible.

Example 3: For $n = 2$. We consider the polynomial,

$$f(x) = \frac{x(x-1)(x-2)(x-3)}{24} \tag{14}$$

We compute the Table 4 with respect to v_2 and v_3 , respectively.

Table 4. The computed results of Equation (15) with respect to v_2 and v_3

$a \in \mathbb{Z}$	g_1	g_2	g_3	g_4	$\sum v_2$
0	∞	0	1	0	∞
1	0	∞	0	1	∞
2	1	0	∞	0	∞
3	0	1	0	∞	∞
4	2	0	1	0	3
$\sum v_3$					
0	∞	0	0	1	∞
1	0	∞	0	0	∞
2	0	0	∞	0	∞
3	1	0	0	∞	∞
4	0	1	0	0	1

$\overline{\sigma_G(I)} = 4$, $\sum_{p \in P} k_p = 4$, $\bar{\alpha} = 3$, $\beta = 2$ and $\gamma = 6$. So, $4 < 4 + 6$. Implies that f is absolutely irreducible.

Example 4: For $n = 5$. We consider the polynomial,

$$f(x) = \frac{x(x-1)(x-2)(x-3)(x-4)}{120}. \tag{15}$$

We compute the Tables 5 and 6 with respect to v_2 , v_3 and v_5 , respectively.

Table 5. The computed results of Equation (16) with respect to v_2 .

$a \in \mathbb{Z}$	g_1	g_2	g_3	g_4	g_5	$\sum v_2$
0	∞	0	1	0	2	∞
1	0	∞	0	1	0	∞
2	1	0	∞	0	1	∞
3	0	1	0	∞	0	∞
4	2	0	1	0	∞	∞
5	0	2	0	1	0	3

Table 6. The computed results of Equation (16) with respect to v_3 and v_5 .

$a \in \mathbb{Z}$	g_1	g_2	g_3	g_4	g_5	$\sum v_3$
0	∞	0	0	1	0	∞
1	0	∞	0	0	1	∞
2	0	0	∞	0	0	∞
3	1	0	0	∞	0	∞
4	0	1	0	0	∞	∞
5	0	0	1	0	0	1
$\sum v_5$						
0	∞	0	0	0	0	∞
1	0	∞	0	0	0	∞
2	0	0	∞	0	0	∞
3	0	0	0	∞	0	∞
4	0	0	0	0	∞	∞
5	1	0	0	0	0	1

f is absolutely irreducible, because $\overline{\sigma_G(I)} = 5$, $\sum_{p \in p} k_p = 5$, $\bar{\alpha} = 3$, $\beta = 3$ and $\gamma = 9$. So, $5 < 5 + 9$.

Example 5: For $n = 6$. We consider the polynomial,

$$f(x) = \frac{x(x-1)(x-2)(x-3)(x-4)(x-5)}{720}. \tag{16}$$

We compute the Tables 7 and 8 with respect to v_2 , v_3 and v_5 , respectively.

Table 7. The computed results of Equation (17) with respect to v_2 , v_3 and v_5 .

$a \in \mathbb{Z}$	g_1	g_2	g_3	g_4	g_5	g_6	$\sum v_2$
0	∞	0	1	0	2	0	∞
1	0	∞	0	1	0	2	∞
2	1	0	∞	0	1	0	∞
3	0	1	0	∞	0	1	∞
4	2	0	1	0	∞	0	∞
5	0	2	0	1	0	∞	∞
6	1	0	2	0	1	0	4
$\sum v_3$							
0	∞	0	0	1	0	0	∞
1	0	∞	0	0	1	0	∞
2	0	0	∞	0	0	1	∞
3	1	0	0	∞	0	0	∞
4	0	1	0	0	∞	0	∞
5	0	0	1	0	0	∞	∞
6	1	0	0	1	0	0	2

Table 8. The computed results of Equation (17) with respect to v_5 .

							$\sum v_5$
0	∞	0	0	0	0	1	∞
1	0	∞	0	0	0	0	∞
2	0	0	∞	0	0	0	∞
3	0	0	0	∞	0	0	∞
4	0	0	0	0	∞	0	∞
5	1	0	0	0	0	∞	∞
6	0	1	0	0	0	0	1

f is absolutely irreducible, because $\overline{\sigma_G(I)} = 7$, $\sum_{p \in P} k_p = 7$, $\bar{\alpha} = 4$, $\beta = 3$ and $\gamma = 12$. So, $7 < 7 + 12$.

4. Conclusion

In conclusion, our adaptation of the Table Method has convincingly demonstrated the absolute irreducibility of binomial polynomials. Through the introduction of a lemma and theorem to elucidate the implications of our method, we have established that binomial polynomials represent a distinctive subset within the realm of polynomials, exhibiting absolute irreducibility within the ring of integer-valued polynomials. Our findings revealed that $\overline{\sigma_G(I)} = \sum_{p \in P} k_p$ and $\gamma \geq 1$, which means that for all $n \geq 2$, and $\overline{\sigma_G(I)} < \sum_{p \in P} k_p + \gamma$, the binomial

polynomials $\binom{x}{n} = \frac{x(x-1)(x-2)\dots(x-n+1)}{n!}$ for $n \geq 1$ is absolute irreducible.

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