

# New Approach to Laplace Differential Transform Method to Solve Nonlinear Partial Differential Equations

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## Abstract

*In this article, we develop a scheme for the Laplace differential transform method (LDTM) to obtain approximate analytical solutions in form of a series of mathematical models of nonlinear partial differential equations. The technique is based on the application of the Laplace transform to nonlinear partial differential equations. The nonlinear term can be handled with the help of the differential transform method. We applied this method to two types of nonlinear Partial differential equations, namely the-Kaup-Kupershmidt equation and generalized Drinfeld-Sokolov equations. The results obtained were compared with exact solutions, which revealed that LDTM is efficient, accurate, reliable and also applicable for solving many other nonlinear Partial differential equations.*

**Keywords:** *Generalized Drinfeld-Sokolov equations, Kaup-Kupershmidt equation, Laplace Differential Transform Method, Nonlinear Partial differential equations.*

## 1. Introduction

The nonlinear partial differential equations play a significant role in fields of applied sciences such as physics, fluid mechanism, bio mathematics and other areas of science and engineering. There are various absorbing and useful features of physical systems concealed in their nonlinear behavior. The analytical solutions of nonlinear partial differential equation are becoming an imperative aspect since it can help us to understand the procedure of the complicated phenomena modeled by nonlinear Partial Differential Equations (PDEs).

As a result, the researchers have shown much interest in the investigation of analytical solution of nonlinear differential equation. Recently, many attentions had been paid to study solutions of nonlinear differential equations by using different methods (we refer to (Adomian, 1984; He and Wu, 2006; Liao, 2003; Wazwaz, 2004; Zhang *et al.*, 2006; Zhou, 1986) and the reference therein). He (2003), in their paper, proposed a new perturbation method which does not require small parameter in an equation. This is different from the Differential Transform method employed by Ali (2017), for obtaining solutions for initial value problems easily convertible to exact. Keskin and Oturanc (2010) presented a general framework of the reduced Differential Transform method for solving Korteweg-de Vries equations.

Furthermore, the methods of Laplace transform and Differential Transform method was combining to form a new scheme and is useful in obtaining solutions of nonlinear differential equations (NDEs) (Keskin and Oturanc, 2010). For the nonlinear models, the Laplace Differential Transform method has shown dependable results derived using (DTM). Keskin and Oturanc (2009), introduced reduced DTM for solving PDEs, giving the properties of the rDTM as a method that helps to overcome demerit of complex calculations of DTMs.

However, previous articles have not considered combination of different PDEs in the application of Laplace Differential Transform method (LDTM). This essentially gives more clarity and credence to the reduced DTMs, which helps solve complex nonlinear problems. In the present article, we apply the Laplace Differential Transform method (LDTM) to two different PDEs namely: Kaup-Kupershmidt equation and generalized Drinfeld-Sokolov

equation. The non-linear terms in the problems were analyzed using Differential transform method and respective iterative algorithms were developed for the determination of the successive terms of series solution in a recursive manner. This paper has been organized as follows: section 2 deals with the analysis of the method. In section 3, we explore the LDTM to solve aforementioned problems of NPDEs. Section 4, conclusions are given.

## 2. Analysis of Laplace Differential Transform Method

The LDTM is a mathematical technique with a broader approach towards solving differential equations by combining the basic principle of Laplace Transform (LT) and Differential Transform method as stated above. The LDTM combines the merits of both LT and DTM to help solve more equations in time space as well as algebraic equations in such domain.

The aim of this section is to discuss the LDTM, let us consider a general form of homogeneous nonlinear equation of the form:

$$Lu(x, t) + Ru(x, t) + Nu(x, t) = f(x, t), \tag{1}$$

where Laplace operator is (denoted by  $\mathcal{L}$ ),  $L$  is an invertible operator,  $R$  is represents the linear differential operator,  $N$  indicates nonlinear term of differential equation, and  $f(x, t)$  is source term. Taking Laplace transform of Equation (1) gives;

$$\mathcal{L}[Lu(x, t)] + \mathcal{L}[Ru(x, t) + Nu(x, t)] = \mathcal{L}[f(x, t)]. \tag{2}$$

where

$$\mathcal{L}[Lu(x, t)] = s^n u(x, t) - \sum_{r=0}^{n-1} s^{n-1-r} f^r(0). \tag{3}$$

Substituting equation (3) into equation (2) and simplifying, gives

$$u(x, t) = \frac{1}{s^n} \mathcal{L}[Ru(x, t) + Nu(x, t)] + \frac{1}{s^n} \mathcal{L}[f(x, t)] + \sum_{r=0}^{n-1} s^{-n} \cdot s^{n-1-r} f^r(0). \tag{4}$$

$$u(x, t) = \frac{1}{s^n} \mathcal{L}[f(x, t)] + \sum_{r=0}^{n-1} s^{-(r+1)} f^r(0) + \frac{1}{s^n} \mathcal{L}[Ru(x, t) + Nu(x, t)]. \tag{5}$$

Introducing the Laplace inverse transform to equation (5), yield.

$$u(x, t) = \mathcal{L}^{-1} \left[ \frac{1}{s^n} \mathcal{L}[f(x, t)] + \sum_{r=0}^{n-1} s^{-(r+1)} \cdot f^r(0) \right] + \mathcal{L}^{-1} \left[ \frac{1}{s^n} \mathcal{L}[Ru(x, t) + Nu(x, t)] \right]. \tag{6}$$

And it can be written as;

$$u(x, t) = g(x, t) + \mathcal{L}^{-1} \left[ \frac{1}{s^n} \mathcal{L}[Ru(x, t) + Nu(x, t)] \right], \tag{7}$$

where  $g(x, t)$  denotes the expression that arises from the given initial condition and the source term which is

$$g(x, t) = \mathcal{L}^{-1} \left[ \frac{1}{s^n} \mathcal{L}[f(x, t)] + \sum_{r=0}^{n-1} s^{-(r+1)} \cdot f^r(0) \right]. \tag{8}$$

The solution will be in the form of infinite series given as

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \tag{9}$$

where the nonlinear term  $N(U_k(x, t))$  are the transformation of the function  $Nu(x, t)$

$$Nu(x, t) = \sum_{k=0}^{\infty} A_k, \tag{10}$$

where  $A_k$  is define in term of Differential Transform Method.

If the function  $w(x, t) = u(x, t)v(x, t)$  the transform is

$$W_k = \sum_{r=0}^k u_r(x)v_{(k-r)}(x).$$

If the function  $w(x, t) = u^2(x, t)$  the transform is

$$W_k = \sum_{r=0}^k u_r(x)u_{(k-r)}(x).$$

Simplifying Equation (8) gives

$$u_0(x, t) = g(x, t). \tag{11}$$

And the recursive relation is given as;

$$u_{k+1} = \mathcal{L}^{-1} \left[ \frac{1}{s^n} \mathcal{L} [Ru_k(x, t) + A_k] \right]. \tag{12}$$

Here  $k = 0,1,2,3, \dots$  and analytical solution  $u(x, t)$  can be approximated by a truncated series

$$u(x, t) = \lim_{N \rightarrow \infty} \sum_{n=0}^N u_n(x, t). \tag{13}$$

### 3. Applications

In this section, we use the LDTM to solve some special cases system of equations and the results of test example are compared with the exact solutions to show the efficiency of the proposed method. Example 1. We consider the nonlinear Kaup-Kupershmidt equation (Gepreel *et al.*, 2011; Mohebbi, 2012):

$$u_t = u_{xxxxx} + 5uu_{xxx} + \frac{25}{2}u_xu_{xx} + 5u^2u_x, \tag{14}$$

with the initial condition.

$$u(x, 0) = -2k^2 + \frac{24k^2}{1 + e^{kx}} - \frac{24k^2}{(1 + e^{kx})^2}, \tag{15}$$

where  $k$  is an arbitrary constant.

Introduction of Laplace transform to (14), we have

$$\mathcal{L}[u_t] = -\mathcal{L}\left[u_{xxxxx} + 5uu_{xxx} + \frac{25}{2}u_xu_{xx} + 5u^2u_x\right]. \tag{16}$$

$$s\mathcal{L}[u] - U(x, 0) = -\mathcal{L}\left[u_{xxxxx} + 5uu_{xxx} + \frac{25}{2}u_xu_{xx} + 5u^2u_x\right]. \tag{17}$$

Simplifying (17) and substituting the initial condition

$$\begin{aligned} \mathcal{L}[u] = \frac{1}{s} & \left\{ -2k^2 + \frac{24k^2}{1 + e^{kx}} - \frac{24k^2}{(1 + e^{kx})^2} \right\} \\ & - \frac{1}{s} \mathcal{L}\left[u_{xxxxx} + 5uu_{xxx} + \frac{25}{2}u_xu_{xx} + 5u^2u_x\right]. \end{aligned} \tag{18}$$

Taking the Laplace inverse of (18), yield

$$\begin{aligned} u(x, t) = -2k^2 + \frac{24k^2}{1 + e^{kx}} - \frac{24k^2}{(1 + e^{kx})^2} \\ - \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[ \frac{\partial^5 u(x)}{\partial x^5} + 5u \frac{\partial^3 u(x)}{\partial x^3} + \frac{25}{2} \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + 5u^2 \frac{\partial u}{\partial x} \right] \right\}. \end{aligned} \tag{19}$$

If we assume the infinite series solutions of the unknown function  $u(x, t)$  of the form

$$u(x, t) = \sum_{k=0}^{\infty} u_k(x, t). \tag{20}$$

Then Equation (19) can easily be written in the form

$$\begin{aligned} u(x, t) = -2k^2 + \frac{24k^2}{1 + e^{kx}} - \frac{24k^2}{(1 + e^{kx})^2} \\ - \mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[ \frac{\partial^5 u(x)}{\partial x^5} + 5A_k(x, t) + \frac{25}{2}B_k(x, t) + 5C_k(x, t) \right] \right\}. \end{aligned} \tag{21}$$

From (19) we write

$$u_0(x, t) = -2k^2 + \frac{24k^2}{1 + e^{kx}} - \frac{24k^2}{(1 + e^{kx})^2}. \tag{22}$$

The recursive relation is given as follow:

$$u_{k+1}(x, t) = -\mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[ \frac{\partial^5 u_k(x)}{\partial x^5} + 5A_k(x) + \frac{25}{2}B_k(x) + 5C_k(x) \right] \right\}, \tag{23}$$

where  $A_k(x)$ ,  $B_k(x)$  and  $C_k(x)$  are represents the transformation of nonlinear in term of Differential Transform method. The first few nonlinear terms are given for the convenience of the reader.

$$\left\{ \begin{aligned} A_k(x) &= u \frac{\partial^3(x)}{\partial x^3} = \sum_{r=0}^k u_r(x) u_{xxx(k-r)}(x) \\ A_0 &= u_0 u_{0xxx} \\ A_1 &= u_1 u_{0xxx} + u_0 u_{1xxx} \\ A_2 &= u_2 u_{0xxx} + u_1 u_{1xxx} + u_0 u_{2xxx}. \end{aligned} \right. \tag{24}$$

$$\left\{ \begin{aligned} B_k(x) &= \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} = \sum_{r=0}^k u_{x(r)}(x) u_{xx(k-r)}(x) \\ B_0 &= u_{0x} u_{0xx} \\ B_1 &= u_{1x} u_{0xx} + u_{0x} u_{1xx} \\ B_2 &= u_{2x} u_{0xx} + u_{1x} u_{1xx} + u_{0x} u_{2xx}. \end{aligned} \right. \tag{25}$$

$$\left\{ \begin{aligned} C_k(x) &= u^2 \frac{\partial u}{\partial x} = \sum_{r=0}^k u_r u_{(k-r)}(x) u_{x(k-r)}(x) \\ C_0 &= u^2_0 u_{0x} \\ C_1 &= 2u_0 u_1 u_{0x} + u^2_0 u_1 \\ C_2 &= 2u_0 u_2 u_{0x} + u^2_1 u_{0x} + 2u_0 u_1 u_{1x} + u^2_0 u_{2x}. \end{aligned} \right. \tag{26}$$

From (23) when  $k = 0$ , we have

$$u_1(x, t) = -\mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[ \frac{\partial^5 u_0(x)}{\partial x^5} + 5A_0(x) + \frac{25}{2} B_0(x) + 5C_0(x) \right] \right\}. \tag{27}$$

Putting the values  $A_0(x)$ ,  $B_0(x)$  and  $C_0(x)$  into (27) then simplifying

$$u_1(x, t) = -\mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L} \left[ \frac{\partial^5 u_0(x)}{\partial x^5} + 5u_0 u_{0xxx} + \frac{25}{2} u_{0x} u_{0xx} + 5u^2_0 u_{0x} \right] \right\}. \tag{28}$$

Follow the same manner

$$u_2(x, t) = -\frac{1452k^{12} (e^{3kx} - 4e^{2kx} + e^{kx}) t^2}{(1 + e^{kx})^4}. \tag{29}$$

$$u_3(x, t) = \frac{5324e^{17} (e^{4kx} - 11e^{3kx} + 11e^{2kx} - e^{kx}) t^3}{(1 + e^{kx})^5}. \tag{30}$$

The approximate analytic series solution is

$$u(x, t) = u[0] + u[1] + u[2] + u[3] \dots \tag{31}$$

$$u(x, t) = -2k^2 + \frac{24k^2}{1 + e^{kx}} - \frac{24k^2}{(1 + e^{kx})^2} - \frac{-264k^7(-e^{kx} + e^{2kx})t}{s(1 + e^{kx})^3} + \frac{1452k^{12}(e^{3kx} - 4e^{2kx} + e^{kx})t^2}{(1 + e^{kx})^4} + \frac{-5324e^{17}(e^{4kx} - 11e^{3kx} + 11e^{2kx} - e^{kx})t^3}{(1 + e^{kx})^5}. \tag{32}$$

The solution is convergent to the exact

$$u(x, t) = -2k^2 + \frac{24k^2}{1 + e^{kx+11k^5t}} - \frac{24k^2}{(1 + e^{kx+11k^5t})^2}. \tag{33}$$

Example 2. Consider the generalized Drinfeld-Sokolov equations (Ugurlu and Kaya, 2008; Afrouzi et al. 2010):

$$\begin{cases} u_t + u_{xxx} - 6(v^\alpha)_x = 0. \\ v_t - 2v_{xxx} + 6uv_x = 0. \end{cases} \tag{34}$$

Subject to initial conditions

$$\begin{cases} u(x, 0) = \frac{-b^2 - 4k^4}{4k^2} + 2k^2 \tanh^2(kx). \\ v(x, 0) = b \tanh(kx). \end{cases} \tag{35}$$

where  $\alpha$  is arbitrary constant. Taking Laplace transform of (34), we have

$$\begin{cases} \mathcal{L}[u_t] = -\mathcal{L}[u_{xxx} - 6uu_x - 6(v^\alpha)_x]. \\ \mathcal{L}[v_t] = -\mathcal{L}[-2v_{xxx} + 6uv_x]. \end{cases} \tag{36}$$

Simplifying (34) and substituting the initial conditions (35) and take the inverse Laplace Transform, we write

$$\begin{cases} u_0(x, t) = \frac{-b^2 - 4k^4}{4k^2} + 2k^2 \tanh^2(kx). \\ v_0(x, t) = b \tanh(kx). \end{cases} \tag{37}$$

The recursive iteration is given as

$$\begin{cases} u_{k+1}(x, t) = -\mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L}[u_{kxxx} - 6A_k - 6B_k] \right\}. \\ v_{k+1}(x, t) = -\mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L}[-2v_{kxxx} + 6C_k] \right\}. \end{cases} \tag{38}$$

where the nonlinear  $A_k, B_k$  and  $C_k$  are transform in term of Differential Transform method in the relation,

$$\begin{cases} A_k = uu_x = \sum_{r=0}^k u_{k-r}(x) \frac{\partial u_r(x)}{\partial x}. \\ A_0 = u_0 \frac{\partial u_0}{\partial x}. \\ A_1 = u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x}. \\ A_2 = u_2 \frac{\partial u_0}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial u_2}{\partial x}. \end{cases} \tag{39}$$

$$\left\{ \begin{aligned} B_k &= (V^\alpha)_x = \frac{\partial v_k^\alpha(x)}{\partial x}. \\ B_0 &= \frac{\partial(v_0^\alpha)}{\partial x}. \\ B_1 &= \frac{\partial(\alpha v_0^\alpha v_1)}{\partial x}. \\ B_2 &= \frac{\partial}{\partial x} \left( \frac{\alpha(\alpha-1)}{2} v_0^{\alpha-2} v_1^2 + \alpha v_0^{\alpha-1} v_2 \right). \end{aligned} \right. \tag{40}$$

$$\left\{ \begin{aligned} C_k &= uv_x = \sum_{r=0}^k u_{k-r}(x) \frac{\partial v_r(x)}{\partial x}. \\ C_0 &= u_0 \frac{\partial v_0}{\partial x} \\ C_1 &= u_1 \frac{\partial v_0}{\partial x} + u_0 \frac{\partial v_1}{\partial x}. \\ C_2 &= u_2 \frac{\partial v_0}{\partial x} + u_1 \frac{\partial v_1}{\partial x} + u_0 \frac{\partial v_2}{\partial x}. \end{aligned} \right. \tag{41}$$

From (38) when  $k = 0$ , we have

$$\left\{ \begin{aligned} u_1(x, t) &= -\mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L}[u_{0xxx} - 6A_0 - 6B_0] \right\}. \\ v_1(x, t) &= -\mathcal{L}^{-1} \left\{ \frac{1}{s} \mathcal{L}[-2v_{0xxx} + 6C_0] \right\}. \end{aligned} \right. \tag{42}$$

Putting the values  $A_0(x)$ ,  $B_0(x)$  and  $C_0(x)$  into (38) (when  $\alpha = 2$ ) (Ugurlu and Kaya, 2008), then simplifying

$$\left\{ \begin{aligned} u_1(x, t) &= \frac{2tk(4k^4 + 3b^2)\sinh(kx)}{\cosh(kx)^3}. \\ v_1(x, t) &= \frac{tb(4k^4 + 3b^2)}{2\cosh(kx)^2k}. \end{aligned} \right. \tag{43}$$

Follow the same manner,

$$\left\{ \begin{aligned} u_2(x, t) &= -\frac{t^2(4k^4 + 3b^2)^2(2\cosh(kx)^2 - 3)}{2\cosh(kx)^4}. \\ v_2(x, t) &= -6t - \frac{4t^2 \sinh(kx)bk^2(4k^4 + 3b^2)(\cosh(kx)^2 - 3)}{\cosh(kx)^5}. \end{aligned} \right. \tag{41}$$

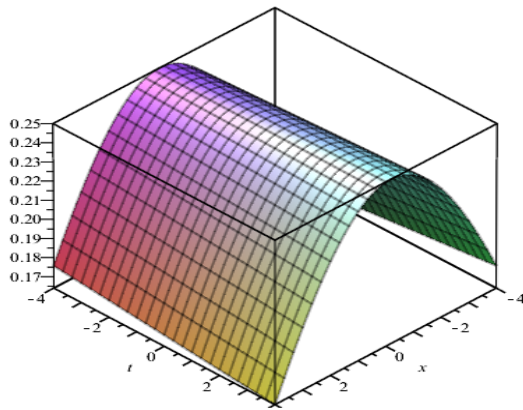
The approximate analytic series solution is

$$\begin{cases} u(x, t) = u[0] + u[1] + u[2] + \dots \\ v(x, t) = v[0] + v[1] + v[2] + \dots \end{cases}$$

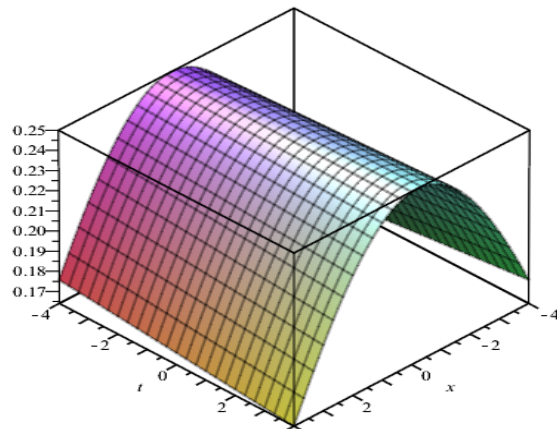
$$\left\{ \begin{aligned} u(x, t) &= \frac{-b^2 - 4k^4}{4k^2} + 2k^2 \tanh^2(kx) + \frac{2tk(4k^4 + 3b^2) \sinh(kx)}{\cosh(kx)^3} - \frac{t^2(4k^4 + 3b^2)^2(2\cosh(kx)^2 - 3)}{2\cosh(kx)^4}. \\ v(x, t) &= bt \tanh(kx) + \frac{tb(4k^4 + 3b^2)}{2\cosh(kx)^2k} - 6t - \frac{4t^2 \sinh(kx)bk^2(4k^4 + 3b^2)(\cosh(kx)^2 - 3)}{\cosh(kx)^5}. \end{aligned} \right. \tag{42}$$

The solution is convergent to the exact

$$\begin{cases} u(x, t) = \frac{-b^2 - 4k^4}{4k^2} + 2k^2 \tanh^2 \left( kx + \frac{3b^2 + 4k^4}{2k} t \right). \\ v(x, t) = b \tanh \left( kx + \frac{3b^2 + 4k^4}{2k} t \right). \end{cases} \quad (43)$$

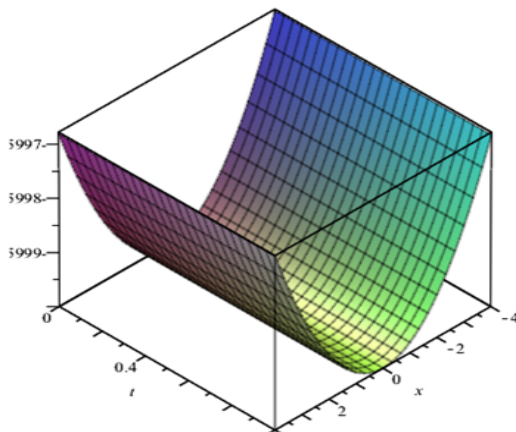


1(a) Approximate solution.

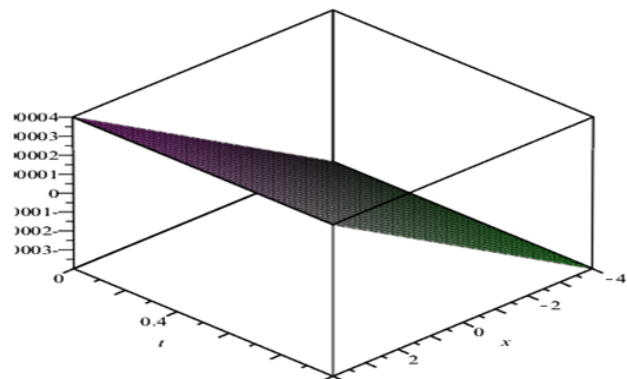


1(b) Exact solution.

Figure 1: Shows the results of (a) approximate solution and (b) the exact solution of nonlinear Kaup-Kupershmidt equations.



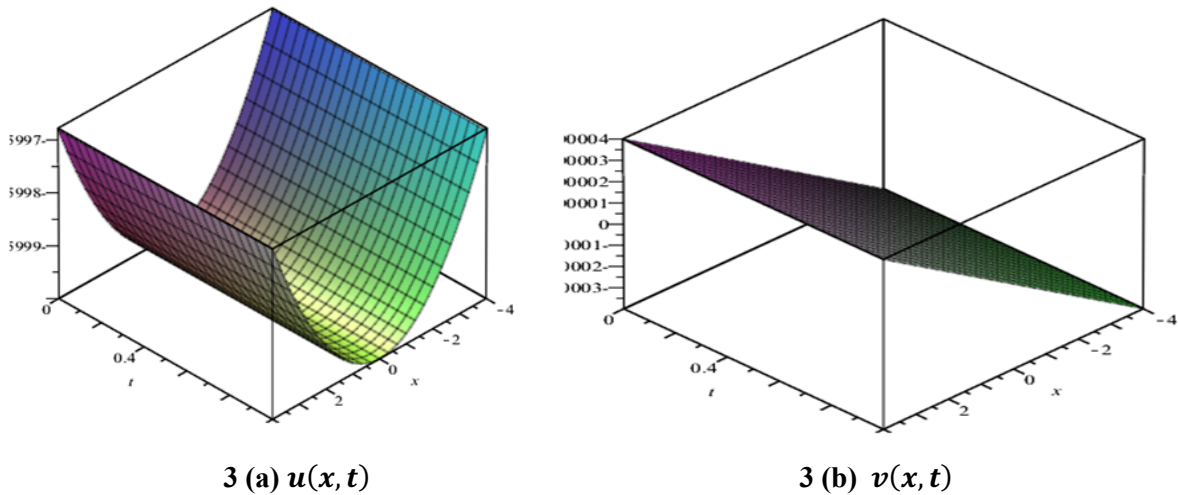
2 (a).  $u(x, t)$



2 (b).  $v(x, t)$

Figure 2: Shows the graph results of the exact solution of generalized Drinfeld-Sokolov (gDs) equations (a)  $u(x, t)$  and (b)  $v(x, t)$  when  $k = 0.01$  and  $b = 0.001$ .





**Figure 3:** Shows the graph results of the approximate solution of generalized Drinfeld-Sokolov (gDs) equations (a)  $u(x, t)$  and (b)  $v(x, t)$  when  $k = 0.01$  and  $b = 0.001$ .

#### 4. Conclusion

The coupling of Laplace Transform method (LTM) and Differential Transform method (DTM) proved very effective and powerful for solving nonlinear partial differential equations as seen from the results obtained. The ability to handle both linear and nonlinear differential equations as well as flexibility in the choice of the differential transform was achieved. Allowing for visualization and customization of the system of equation. It is clear that the method yields very reliable and accurate approximate solutions with only few iterates, which is efficient for computational implementation and solving problems that needs analytical solutions. Hence, the conclusion drawn through this work is that LDTM can also be applied for solving other nonlinear partial differential equations, due to the efficiency and flexibility in the application to get the appropriate results which are useful in physics, engineering and mathematical biology.

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