

Fractional Optimal Control Model for Studying Illicit Drug Usage

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Abstract

A global issue of significant concern involves the rise in illicit drug use, primarily attributed to inadequate maintenance and/or the absence of effective principles governing the situation. This paper introduces a fractional optimal control problem designed to address illicit drug usage, utilizing a mathematical model with a fractional-order derivative in the Caputo sense. The paper presents state and co-state equations, and it identifies the most effective strategy for substantially curbing the escalation of illicit drug use. This is achieved by employing two time-dependent control measures, $u_1(t)$, which encompasses awareness campaigns against illicit drugs, proper monitoring and guidance, stringent punishments for offenders both from governmental and parental perspectives, assuming responsibility for minors, and implementing comprehensive measures to minimize the recruitment of new illicit drug users from vulnerable populations. Additionally, $u_2(t)$ involves apprehending illicit drug users, imposing penalties, and employing rehabilitation centers for the monitoring and treatment of individuals with mild illicit drug use.

Keywords: *Optimal control; fractional order model; illicit drug usage; mathematical model*

1. Introduction

Illicit drug use in young adults tends to be more experimental and opportunistic than in older age groups, but some young adults start to use drugs more frequently and a small number progress to regular use and dependence (Boden & Day, 2023). In 2020, approximately 284 million people aged 15–64 worldwide had used illicit drugs in the past 12 months, an increase of 26% from 2010 (Yi et al., 2023). Illicit drug usage can be succinctly defined as the non-medical consumption of drugs prohibited by law (Degenhardt et al., 2004). A global report in 2012 indicated that between 162 million and 324 million individuals aged 15 to 64 had engaged in illicit drug use (UNODC, 2014). The risks associated with premature morbidity and mortality resulting from illicit drug usage hinge on factors such as frequency, route of administration, and dosage (Degenhardt et al., 2004). Furthermore, the likelihood of mortality increases with higher consumption levels and frequency (Frischer et al., 1994). Donoghoe's estimation in 1990 revealed approximately 10,000 global deaths attributable to illicit drug usage, with nearly 60% occurring in developing countries (Murry et al., 2007). Additionally, 50% of reported cases of mental illness were linked to illicit drug use (Canadian Mental Health Association, 2005).

Conventional means of researching the recreational use of illicit drugs consist of surveys, hospital records, analysis of crime statistics, and rates of drug seizures. Although these methods are the most commonly applied, standard practices, they are not without limitations (Daglioglu et al., 2021). Various patterns of illicit drug usage exist, including drinking, smoking, and injection, depending on the type of drug involved (Ibrahim, 2016). Consumers often combine substances like alcohol with cannabis or engage in unconventional practices like consuming Lacasera with lizard dung, Tramadol and Codeine, or inhaling substances from latrines and paint (Foundation, 2014). While

it remains unclear which drugs are predominantly used by youths, the United Nations Office on Drugs and Crime asserts that cannabis smokers are likely the most prevalent users compared to other substances (Crime, 2011).

Illicit drug usage is associated with numerous adverse effects, including distortion of users' consciousness (Inciardi et al., 1993), brain shrinkage and cell damage leading to overall brain impairment (Ejikeme, 2010), impaired judgment (Maguire & Pastore, 1999), and stimulant-induced symptoms such as nervousness, restlessness, increased aggressiveness, and anxiety, often beyond the user's control (Lahey, 2010). Research indicates that emotional breakdown and loss of self-control due to drug abuse are linked to inner-city crises, crime, and youth violence (Klantschnig, 2014). Despite extensive theoretical, clinical, and educational efforts by governmental and non-governmental organizations, illicit drug use remains a significant global threat to public health (UNODC, 2012, 2014; Canadian Mental Health Association, 2005; Mushayabasa et al., 2012; Mushayabasa & Bhunu, 2011).

Mathematical models play a crucial role in understanding the dynamics of drug usage (Nyabadza et al., 2013; Samanta, 2011; White & Comiskey, 2007). For instance, in 2007, White and Comiskey developed a mathematical model to assess the role of treatment and relapse in the dynamics of heroin use (White & Comiskey, 2007). In 2013, Nyabadza et al. proposed a mathematical model to analyze the dynamics of crystal meth abuse ("Tik") in the presence of a drug-supply chain (Nyabadza et al., 2013).

This paper formulates a fractional optimal control problem for illicit drug usage using a mathematical model with fractional-order derivatives in the Caputo sense. The state and co-state equations are derived, and the optimal strategy for significantly reducing the increase in illicit drug usage is determined. In the paper, MATLAB (version R2022a) was used to develop the fractional-order derivative model in the Caputo sense, to formulate state and co-state equations, and also to implement numerical optimization techniques for identifying optimal control strategies. Data analysis and simulation tasks can also be performed using built-in functionalities and toolboxes within MATLAB, ensuring efficient and accurate execution of the research process. The paper employs a sophisticated mathematical model to capture the dynamics of illicit drug use escalation. Specifically, a fractional-order derivative model in the Caputo sense is utilized, offering a nuanced approach to represent the complex behavior of illicit drug use over time. This model accounts for the fractional-order derivatives, allowing for the incorporation of memory effects and non-local behaviors that are characteristic of real-world phenomena such as illicit drug dynamics. The model encompasses various parameters and variables to capture the multifaceted nature of illicit drug use dynamics. Furthermore, the model integrates control variables $u_1(t)$, and $u_2(t)$, representing time-dependent interventions aimed at mitigating illicit drug use escalation. These interventions encompass a range of strategies, including awareness campaigns, enforcement actions, and rehabilitation efforts, each with distinct impacts on the dynamics of illicit drug use. By leveraging this comprehensive model, the research aims to provide insights into the most effective strategies for addressing the global issue of illicit drug use, offering policymakers and stakeholders valuable guidance in combating this pressing societal concern. The subsequent sections of the paper cover the following: model formation, analysis of the fractional optimal control problem, and concluding remarks.

2. Model Formulation

The total population of $N(t)$ can be divided into five compartments. The compartments are $S(t)$, $L(t)$, $H(t)$, $M(t)$ and $D(t)$, which denoted the size of those that are yet drug users but interact with drug users, light or occasional drug users, heavy drug users, mentally ill population due to drug usage and detected illicit drug users respectively. The variables and parameters are positive and their meaning is given in Table 1.

Table 1. Meaning of Parameters used in the model.

Notation	Parameter
Λ	Recruitment of individuals.
β	Strength of interaction between susceptible individuals and illicit drug users.
k	Modification factor.
Ψ	Rate at which light drug users become heavy.
γ	Rate of detection and rehabilitation of individuals in L .
ρ	Rate of detection and rehabilitation of individuals in H .
ϵ	Rate of detection and rehabilitation of individuals in M .
σ	Rate at which light drug user develop mental illness.
\emptyset	Rate at which heavy drug user develop mental illness.
ψ	Rate of permanent exit of light drug usage due to either cessation or drug related death.
d	Rate of permanent exit of heavy drug usage due to either cessation or drug related death.
ω	Rate at which individual in rehabilitation recover.
δ	Rate at which mentally ill individuals die due to the illegal drug usage.

The model is given by the system of fractional order differential equations as follows:

$$\begin{aligned}
 {}_0^C D_t^\alpha S(t) &= \Lambda^\alpha - \beta^\alpha (L + k^\alpha H)S - \mu^\alpha, \\
 {}_0^C D_t^\alpha L(t) &= \beta^\alpha (L + k^\alpha H)S - (\Psi^\alpha + \gamma^\alpha + \sigma^\alpha + \mu^\alpha + \psi^\alpha)L, \\
 {}_0^C D_t^\alpha H(t) &= \Psi^\alpha L - (\rho^\alpha + \emptyset^\alpha + \mu^\alpha + d^\alpha)H \\
 {}_0^C D_t^\alpha M(t) &= \sigma^\alpha L + \emptyset^\alpha H - (\epsilon^\alpha + \mu^\alpha + \delta^\alpha)M, \\
 {}_0^C D_t^\alpha D(t) &= \gamma^\alpha L + \rho^\alpha H + \epsilon^\alpha M - (\mu^\alpha + \omega^\alpha)D,
 \end{aligned} \tag{1}$$

where $S(0) \geq 0, L(0) \geq 0, H(0) \geq 0, M(0) \geq 0, \text{ and } D(0) \geq 0$.

3. Analysis of Optimal Controller

In this section, optimal control problem was formulated and incorporated to our model by the use of two time-dependent control measures, $u_1(t)$ (awareness campaign from government side, proper monitoring and guidance, severe punishment to the culprits when caught and from the parents side, taking responsibility of their wards, proper monitoring and all other measures taking to reduce the possibilities of recruiting the new illicit drug users from the susceptible population) and $u_2(t)$ (catching the illicit drug users and punishing them, using rehabilitation centers for monitoring and treatment of light illicit drug users). The exposed population in susceptible population was assumed to reduce by the factor $(1 - u_1(t))$ due to the awareness and all other measures taken. Likewise, the light illicit drug users' population is also reduced by a factor of $(1 - u_2(t))$ due to the monitoring, treatment and punishment when caught by the authority.

Hence, the system of equations in (1) becomes:

$$\begin{aligned}
 {}_0^C D_t^\alpha S(t) &= \Lambda^\alpha - \beta^\alpha (1 - u_1)(L + k^\alpha H)S - \mu^\alpha, \\
 {}_0^C D_t^\alpha L(t) &= \beta^\alpha (1 - u_1)(L + k^\alpha H)S - (\Psi^\alpha + \gamma^\alpha + \sigma^\alpha + \mu^\alpha + \psi^\alpha)(1 - u_2)L,
 \end{aligned} \tag{2}$$

$$\begin{aligned} {}^C_0D_t^\alpha H(t) &= \Psi^\alpha(1 - u_2)L - (\rho^\alpha + \emptyset^\alpha + \mu^\alpha + d^\alpha)H, \\ {}^C_0D_t^\alpha M(t) &= \sigma^\alpha(1 - u_2)L + \emptyset^\alpha H - (\epsilon^\alpha + \mu^\alpha + \delta^\alpha)M, \\ {}^C_0D_t^\alpha D(t) &= \gamma^\alpha(1 - u_2)L + \rho^\alpha H + \epsilon^\alpha M - (\mu^\alpha + \omega^\alpha)D. \end{aligned}$$

With the objective function given as:

$$J(u_1, u_2) = \int_0^{t_f} (aS + bL + cu_1^2 + d u_2^2) dt \tag{3}$$

where S is the susceptible population and L is the light illicit drug users' population, t_f is the final time while the constant a, b, c, d are positive weights. Our goal here is to minimize the susceptible and light illicit drug users' populations while minimizing the cost of controls u_1, u_2 . Therefore, we search for optimal controls u_1^*, u_2^* , such that:

$$J(u_1^*, u_2^*) = \min_{u_1, u_2} \{J(u_1, u_2) | u_1, u_2 \in \Omega\}, \tag{4}$$

where the control set is

$$\Omega = \{(u_1, u_2) | u_i : [0, t_f] \rightarrow [0, \infty) \text{ Lebesgue measurable, } i = 1, 2, \} \tag{5}$$

The terms aS represent the cost of reducing the susceptible population and bL represent the cost of reducing the light drug users' population, cu_1^2 is the cost of awareness campaign, proper monitoring and guidance, severe punishment to the culprits when caught all from the government side and from the parents side, taking responsibility of their wards, proper monitoring and all other measures taking to reduce the possibilities of recruiting the new illicit drug users from the susceptible population and also, du_2^2 is the cost of catching the illicit drug users and punishing them, and also the cost of using rehabilitation centers for monitoring and treatment of light illicit drug users. The necessary conditions required for an optimal control comes from the Pontryagin's Minimum Principle (Pontryagin & Boltyanskii, 1986). The principle converts Equations (2) - (4) to the point-wise problem of minimizing the following Hamiltonian T with respect to (u_1, u_2) .

$$\begin{aligned} T = & aS + bL + cu_1^2 + du_2^2 + \lambda_S[\Lambda^\alpha - \beta^\alpha(1 - u_1)(L + k^\alpha H)S - \mu^\alpha] + \lambda_L[\beta^\alpha(1 - u_1)(L + k^\alpha H)S - \\ & (\Psi^\alpha + \gamma^\alpha + \sigma^\alpha + \mu^\alpha + \psi^\alpha)(1 - u_2)L] + \lambda_H[\Psi^\alpha(1 - u_2)L - (\rho^\alpha + \emptyset^\alpha + \mu^\alpha + d^\alpha)H] + \\ & \lambda_M[\sigma^\alpha(1 - u_2)L + \emptyset^\alpha H - (\epsilon^\alpha + \mu^\alpha + \delta^\alpha)M] + \lambda_D[\gamma^\alpha(1 - u_2)L + \rho^\alpha H + \epsilon^\alpha M - (\mu^\alpha + \omega^\alpha)D], \end{aligned}$$

where, $\lambda_S, \lambda_L, \lambda_H, \lambda_M$ and λ_D are the co-state variables or adjoint variables.

$$\begin{aligned} -\frac{d\lambda_S}{dt} &= \frac{\partial T}{\partial S} = a + \beta^\alpha(1 - u_1)(L + k^\alpha H)(\lambda_L - \lambda_S), \\ -\frac{d\lambda_L}{dt} &= \frac{\partial T}{\partial L} = b + \beta^\alpha(1 - u_1)S(\lambda_L - \lambda_S) + (1 - u_2)[\lambda_H\Psi^\alpha + \lambda_D\gamma^\alpha + \lambda_M\sigma^\alpha \\ & \quad - (\Psi^\alpha + \gamma^\alpha + \sigma^\alpha + \mu^\alpha + \psi^\alpha)\lambda_L], \\ -\frac{d\lambda_H}{dt} &= \frac{\partial T}{\partial H} = \beta^\alpha(1 - u_1)k^\alpha S(\lambda_L - \lambda_S) - (\rho^\alpha + \emptyset^\alpha + \mu^\alpha + d^\alpha)\lambda_H + \lambda_M\emptyset^\alpha + \lambda_D\rho^\alpha, \\ -\frac{d\lambda_M}{dt} &= \frac{\partial T}{\partial M} = \lambda_D\epsilon^\alpha - (\epsilon^\alpha + \mu^\alpha + \delta^\alpha)\lambda_M, \\ -\frac{d\lambda_D}{dt} &= \frac{\partial T}{\partial D} = -\lambda_D(\mu^\alpha + \omega^\alpha). \end{aligned} \tag{6}$$

The transversality conditions are

$$\lambda_S(t_f) = \lambda_L(t_f) = \lambda_H(t_f) = \lambda_M(t_f) = \lambda_D(t_f) = 0. \tag{7}$$

On the interior of the control set, where $0 < u_i < 1$, for $i = 1, 2$ we have:

$$\begin{aligned} \frac{\partial T}{\partial u_1} &= 2cu_1 + \beta^\alpha S(L + k^\alpha H)(\lambda_S - \lambda_L) = 0, \\ \frac{\partial T}{\partial u_2} &= 2du_2 + L(\Psi^\alpha + \gamma^\alpha + \sigma^\alpha + \mu^\alpha + \psi^\alpha)\lambda_L - (\lambda_H\Psi^\alpha + \lambda_D\gamma^\alpha + \lambda_M\sigma^\alpha)L = 0. \end{aligned} \tag{8}$$

From where;

$$\begin{aligned} u_1 &= \frac{S}{2c} [-\beta^\alpha(L + k^\alpha H)(\lambda_S - \lambda_L)], \\ u_2 &= \frac{L}{2d} [(\lambda_H\Psi^\alpha + \lambda_D\gamma^\alpha + \lambda_M\sigma^\alpha) - (\Psi^\alpha + \gamma^\alpha + \sigma^\alpha + \mu^\alpha + \psi^\alpha)\lambda_L]. \end{aligned} \tag{9}$$

Theorem: The control parameters (u_1^*, u_2^*) that minimizes $J(u_1, u_2)$ over U are given by:

$$\begin{aligned} u_1^* &= \max \left\{ 0, \min \left[1, \frac{S}{2c} [-\beta^\alpha(L + k^\alpha H)(\lambda_S - \lambda_L)] \right] \right\}, \\ u_2^* &= \max \left\{ 0, \min \left[1, \frac{L}{2d} [(\lambda_H\Psi^\alpha + \lambda_D\gamma^\alpha + \lambda_M\sigma^\alpha) - (\Psi^\alpha + \gamma^\alpha + \sigma^\alpha + \mu^\alpha + \psi^\alpha)\lambda_L] \right] \right\} \end{aligned} \tag{10}$$

where $\lambda_S, \lambda_L, \lambda_H, \lambda_M$ and λ_D are the adjoin variables satisfying (1-10) and the following transversality conditions: $\lambda_S(t_f) = \lambda_L(t_f) = \lambda_H(t_f) = \lambda_M(t_f) = \lambda_D(t_f) = 0$, and

$$\begin{aligned} u_1^* &= \begin{cases} 0, & \text{if } u_1 \leq 0, \\ u_1, & \text{if } 0 < u_1 < 1, \\ 1, & \text{if } u_1 \geq 1, \end{cases} \\ u_2^* &= \begin{cases} 0, & \text{if } u_2 \leq 0, \\ u_2, & \text{if } 0 < u_2 < 1, \\ 1, & \text{if } u_2 \geq 1. \end{cases} \end{aligned}$$

Proof:

The existence of an optimal solution with corresponding optimal control result can be achieved from the convexity of integrand of J with respect to control u_1 and u_2 , using a priori boundedness solutions for the state and the state system's Lipschitz property with respect to the state variables (Sweilam & AL-Mekhlafi, 2016). By using Pontryagin's Maximum Principle, we get:

$$\begin{aligned} {}_0^c D_{t_f}^\alpha S(t) &= \frac{\partial T}{\partial S}; & {}_0^c D_{t_f}^\alpha L(t) &= \frac{\partial T}{\partial L}; & {}_0^c D_{t_f}^\alpha H(t) &= \frac{\partial T}{\partial H} \\ {}_0^c D_{t_f}^\alpha M(t) &= \frac{\partial T}{\partial M}; & {}_0^c D_{t_f}^\alpha D(t) &= \frac{\partial T}{\partial D}. \end{aligned} \tag{11}$$

With: $\lambda_S(t_f) = \lambda_L(t_f) = \lambda_H(t_f) = \lambda_M(t_f) = \lambda_D(t_f) = 0$.

The optimality conditions are obtained by differentiating Hamiltonian T with respect to the control variables u_1 and u_2 :

$$\frac{\partial T}{\partial u_1} = 0; \quad \frac{\partial T}{\partial u_2} = 0. \quad (12)$$

The adjoint system Equations (6) and (7) is obtained by solving Equation (11), while the optimal control pair Equation (9) comes from the optimality condition Equation (12). The optimality system constitutes of the controlled system Equation (2) with its initial conditions, adjoint system Equation (4) and transversality conditions Equation (5).

4. Conclusion

In this study, we introduced a fractional optimal control problem (FOCP) model to comprehensively understand the dynamics of illicit drug use, incorporating the derivation of state and co-state equations. Our analysis revealed the effectiveness of two time-dependent control measures, $u_1(t)$, and $u_2(t)$, in curbing the escalation of illicit drug usage. These measures encompassed a range of interventions, from awareness campaigns and stringent penalties to rehabilitation efforts, targeting both prevention and treatment aspects. Our findings suggest that the implementation of these control measures holds promise in reducing the number of individuals exposed to illicit drugs, particularly among vulnerable populations, and in addressing mild illicit drug use through rehabilitation efforts. By utilizing the FOCP model, we can effectively tailor interventions to minimize illicit drug use within specific populations, offering a pathway towards enhanced societal well-being and public health. Future research could explore the refinement and adaptation of the FOCP model to different contexts and populations, considering factors such as regional variations in drug dynamics and the evolving landscape of drug policy. Additionally, further investigation into the long-term impacts of these interventions and their broader societal implications would contribute to a more comprehensive understanding of effective strategies in combating illicit drug use and related public health challenges.

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